Theoretical and Mathematical Physics

Dmitri Fursaev
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# Operators, Geometry and Quanta

## Methods of Spectral Geometry in Quantum Field Theory

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*We dedicate this book to our parents*

# Preface

Massive penetration of operators in physics started with the birth of Quantum Mechanics. The mathematical methods used in quantum theory, being rather elementary at the very beginning, rapidly achieved a high level of sophistication. It was realized soon that relevant information on the operators may be collected in the objects called the *spectral functions*, which depend on the operator in question and on a real or complex parameter. In the context of quantum field theory (QFT) the most frequently used spectral function is the *heat kernel*. This function was first applied to the problems of quantum physics by Fock [111] already in 1937, and then by Schwinger in his seminal paper [225]. In 1960's DeWitt [77–80] put the heat kernel as a corner stone for his method of calculating quantum corrections. About 10 years later, Dowker and Critchley [92] and Hawking [155] suggested a regularization scheme of quantum field theory based on another prominent spectral function, the *zeta-function*. All these works determined for the rest of the century a landscape of methods of quantum field theory based on geometrical properties of operators. This has led to very interesting and important developments in practically all areas of QFT, ranging from quantum gravity, to anomalies, strings, and the Casimir effect. Apart form powerful technical tools this approach provided the physicists with a new rather adequate language to describe complicated phenomena, and with a new, very fruitful, point of view on quantum effects in general.

The main idea of the approach is easy to understand even without knowing precise definitions. Consider a QFT on some curved space. The spectrum of quantum fluctuations is defined by the spectrum of an operator, say, a Laplacian, on this space. The aim then is two-fold. On one hand, it is to relate quantities of interest in QFT, such as a ground state energy, to functions of the spectrum of this operator, i.e., to spectral functions. On the other hand, one looks for a connection of the spectral functions to geometric characteristics of the space. It is desirable to have sufficiently general relations operating with essential ingredients of the problem, such as geometric invariants of the manifold, and not depending on inessential features, for instance, on a particular choice of coordinates.

Interestingly, similar problems were being solved in mathematics at about the time. Kac [165] put the problem in the following way: Can one hear the shape of a drum? In other words, knowing the acoustic eigenfrequencies, what can one say about the geometry? Probably the first step in this direction was done long ago by H. Weyl [257, 258] who found a relation between asymptotic distribution of eigenvalues of a Laplacian and the volume of the manifold. Minakshisundaram and Pleijel [189] derived more detailed relations involving other geometric invariants. A firm basis for such kind of calculations was developed by Seeley [226–228], and a powerful technique for actual computations was suggested by Gilkey [132].

By mid 1980's the methods related to spectral functions, mainly to the heat kernel, became standard in QFT, especially in curved space, but also in all cases when non-trivial geometry and topology were essential. The famous Birrell and Davies book [37] could be found on the desk of practically everyone even remotely connected with quantum gravity, often accompanied by a more technical review by Barvinsky and Vilkovisky [26]. New challenges appeared very soon. Among them there were spectral problems with boundaries stemming from quantum cosmology, strings, and the Casimir effect. Then appeared problems with various types of singularities following from the brane world scenario and black hole physics. Finally, the 21st century put spectral problems on noncommutative spaces in the center of interest.

Of course, many good books appeared meanwhile. Some of them are listed below as recommended literature. There is, however, a gap, which we would like to fill in by the present work. We were aiming at writing a text starting with the level of an advanced textbook, i.e., containing all basic information, especially on the mathematical side, and gradually reaching rather advanced physical topics. We tried to make the book as selfcontained as possible to be useful for both active researchers and graduate students. Inclusion of more than a hundred exercises with their solutions makes it possible to use this material in lecture courses on physical applications of the spectral theory.

These aims determined the choice of the material and the style of the presentation. The exposition of main mathematical methods is very detailed, though not always reaching the depth and generality of specialized research monographs. In applications, instead of studying one particular area in all detail, we took examples from various fields, including finite temperature field theory, anomalies, quantum solitons, strings, and noncommutative field theories. In each case, we demonstrate how the use of general methods allows to achieve interesting and important results in an elegant and relatively easy manner. All applications are taken from active areas of research. We organized this material to prepare the reader to work further on his/her own in any specific area of QFT.

This book is organized as follows. Part I contains some basic information and serves to settle notations, but not only. Chapter 1 devoted to differential geometry, contains some less standard material on boundaries and singularities. Chapter 2 introduces main notions of QFT basing on relativistic inner products rather than on usual operator quantization. This facilitates applications to the problems in the rest of this book and, in particular, is more convenient in relation to free fields theories in classical backgrounds.

Part II is devoted to mathematical foundations, namely, to the spectral geometry. Chapter 3 explains main properties of operators. Chapters 4, 5 are the central Chapters for this book. Chapter 4 is an introduction to the heat equation and asymptotic properties of the heat kernel expansion. It is organized so that to present briefly a variety of techniques for computation of the heat coefficients on different base manifolds. Chapter 5 contains definitions of main spectral functions, lists their properties, and methods of computation. It defines zeta-functions and determinants of differential operators, explains their transformation properties and the merit of the index theorem. Much space is devoted to variations of the determinants, which will later serve as a basis for calculations of quantum anomalies. Chapter 6 deals with non-linear spectral problems, for which the "eigenvalues" enter the operator itself.

Part III contains applications to various problems in physics. The chapters in this part are relatively independent, except Chap. 7 which introduces the effective action, a notion used many times later on. We use the spectral geometry methods to reproduce a number of known QFT results which are derived usually with the help of Feynman diagrams. Among them are one-loop effective potential and beta functions in gauge theories. In Chap. 8 we turn to the quantum anomalies and calculate almost all known types of anomalies, including gravitational and parity anomalies, for two dimensional models. In Chap. 9 we consider the methods of calculations of the vacuum energy, with the quantum corrections to the kink mass being the principal example. Applications to string theory are contained in Chap. 10 where we derive the Born-Infeld action for open strings and come to noncommutativity of the coordinates of string endpoints. Chapter 11 is devoted to spectral geometry and field theory on noncommutative manifolds, which is studied by using the same universal tools.

Each chapter contains exercises. Some of them are included for pure pedagogical reasons, others are interesting as a complementary material. In any case, exercises are an integral part of this book. We encourage the reader at least to look at their formulations. Solutions to all exercises are given in Part IV.

Not to distract the attention of the reader we avoided references in the main text unless absolutely necessary. Instead, we added sections with literature remarks at the end of each chapter. Because of a vast volume of material we were not able to mention all relevant references. Instead we tried to give a starting point for a literature search.

Here we like to mention several general sources. For more mathematics we recommend the monographs by Gilkey [133, 134] and the one by Kirsten [169], which contains also an analysis of physical applications. The review paper [243] gives an overview of the heat kernel methods in QFT. There are numerous research monographs treating various aspects of applications of spectral functions to QFT and quantum gravity [18, 53, 100, 101, 103–105]. A recent elementary introduction into quantum physics in curved spaces including some of the heat kernel methods is [194]. A very detailed discussion of quantum anomalies may be found in the book by Bertlmann [35]. For a long time the zeta-function techniques were applied to calculation of the Casimir effect. Modern status of this area is described in [45, 187].

We benefited from numerous discussions with our friends and colleagues, and we are grateful to all of them. Special thanks go to J. Buchbinder and H. Grosse for a number of valuable remarks. This work was made possible due to generous support from various grant agencies. In particular, D.V.V. was supported by DFG while in Germany, and by FAPESP and CNPq while in Brazil.

# Contents

## Part IV Problem Solving

# Notation Index[1]

$\mathscr{M}$: manifold
$n$: dimensionality of $\mathscr{M}$
$\partial\mathscr{M}$: boundary of $\mathscr{M}$
$g_{\mu\nu}$: metric tensor on $\mathscr{M}$, $g = \det g_{\mu\nu}$
$\mathbb{R}^{n_-,n_+}$: pseudo-Euclidean space with the signature $(n_-, n_+)$
signature of the metric on Lorentzian manifolds is $(1, 3)$, the time coordinate corresponds to a negative component of the metric
$\Gamma^\lambda_{\mu\nu}$: Christoffel connection (1.4)
$[A, B] = AB - BA$, $\{A, B\} = AB + BA$
$\nabla_\mu$: covariant derivative, usually the one which annihilates the metric, $\nabla_\mu g_{\nu\rho} = 0$
components of the Riemann tensor $R^\sigma{}_{\rho\mu\nu}$ are defined in (1.9), (1.10) as

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)V_\rho = -V_\sigma R^\sigma{}_{\rho\mu\nu},$$

$$R^\lambda{}_{\mu\nu\kappa} = \partial_\nu\Gamma^\lambda_{\mu\kappa} - \partial_\kappa\Gamma^\lambda_{\mu\nu} + \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta}$$

Ricci tensor, (1.15), scalar curvature, (1.16), are, respectively

$$R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}, \qquad R = g^{\mu\nu}R_{\mu\nu}.$$

$\tilde{\varepsilon}^{\mu_1\mu_2\ldots\mu_n}$: totally antisymmetric Levi-Civita symbol of rank $n$, $\tilde{\varepsilon}^{12\ldots n} = 1$
$\varepsilon^{\mu_1\mu_2\ldots\mu_n}$: Levi-Civita tensor of rank $n$, $\varepsilon^{\mu_1\mu_2\ldots\mu_n} = g^{-1/2}\tilde{\varepsilon}^{\mu_1\mu_2\ldots\mu_n}$
$\chi[\mathscr{M}]$: Euler characteristic of $\mathscr{M}$
$C_\beta$: two-dimensional cone (1.96)
$\mathscr{E}$: fiber bundle over $\mathscr{M}$ with the fiber $\mathscr{F}$, locally $\mathscr{E} = \mathscr{M} \times \mathscr{F}$
$e^\mu_a$: orthonormal frame (1.44)
$w^{ab}_\mu$: Levi-Civita connection (1.51)

[1]Here we list main notations in the order of their appearance in the text.

another expression for the Riemann curvature tensor (1.52):

$$R^{ab}{}_{\mu\nu} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\nu{}^{ab} + \omega_\mu{}^a{}_c \omega_\nu{}^{cb} - \omega_\mu{}^a{}_c \omega_\mu{}^{cb}$$

$\gamma^\mu$: Dirac gamma-matrices (1.55)

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\nu = 2g^{\mu\nu}$$

$\gamma^a = \gamma^\mu e^a_\mu$
$\gamma_*$: chirality matrix defined in (1.59) for even dimension $n$
$\bar{\psi} = \psi^\dagger \beta$: Dirac conjugated spinor, $\beta \equiv i\gamma^{a=0}$, $\beta^2 = 1$
$I[\varphi, \phi]$: classical action, $\varphi$ is a set of dynamical variables, $\phi$ is a set of background fields

classical stress-energy tensor (1.22):

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta I}{\delta g_{\mu\nu}}$$

$\Sigma$: co-dimension $k$ hypersurface in $\mathscr{M}$, $(n_i)_\mu$ normals to $\Sigma$, $i = 1, \dots, k$. Normal vector is assumed to be inward pointing if $\Sigma = \partial\mathscr{M}$ is a boundary of $\mathscr{M}$
$(K_i)_{\mu\nu}$: extrinsic curvatures of $\Sigma$, if $h^\alpha_\mu$ tensor which projects on the tangent space to $\Sigma$, then

$$(K_i)_{\mu\nu} = -h^\alpha_\mu h^\beta_\nu \nabla_\beta n_{i\alpha}$$

Extrinsic curvature of the boundary is defined in (1.87)
$\langle f_1, f_2 \rangle$: relativistic product (2.9), $f_1$, $f_2$ are two solutions to a wave equation
$\Im f$, $\Re f$ are, respectively, imaginary and real parts of a complex quantity $f$
$\mathrm{L}^2$: Hilbert space of square integrable functions on $\mathscr{M}$ or of square integrable section of $\mathscr{E}$
$(f_1, f_2)$: scalar product in $\mathrm{L}^2$
$L$: second order elliptic operator on $\mathrm{L}^2$, see (3.1) and (3.2)
$\not{D}$: Dirac type operator, see (3.6) and (3.7)
$\zeta_R(s, a)$: generalized Riemann zeta function (5.4)
$\zeta(s; L)$: zeta function of an elliptic operator $L$
$K(x, y|t)$: heat kernel, see (4.3)–(4.5)
tr: trace over bundle indices
$\mathrm{Tr}_{\mathrm{L}^2}$: functional trace on $\mathrm{L}^2$
$K(Q, L; t)$: heat trace for an operator $L$, (4.6), $Q$ is a partial differential operator
$K(L; t) \equiv K(1, L; t)$
$K(f, L; t)$: smeared heat trace (4.9) for an operator $L$ and a test function $f$
$a_p(f, L)$: heat kernel coefficients for the smeared trace, see (4.9)
$a_p(Q, L)$: heat kernel coefficients for $K(Q, L; t)$, expressions for $a_p(Q, L)$ are listed in (4.124) when $Q$ is a matrix valued function
$\rho(\lambda)$: spectral density, see (5.37)
$N(\lambda)$: counting function, see (5.38)
$\rho_\alpha(\lambda)$: smeared spectral density (the Riesz means of the spectral density $\rho(\lambda)$), see (5.40)
$W[\phi]$: (one loop) effective action

# Part I
# The Basics

# Chapter 1
# Geometrical Background

## 1.1 Fields and Particles

Experimental data available at present tell us that the high energy physics should be formulated as a quantum field theory. In accelerator experiments, one detects particles which are excitations of fields characterized by certain energy, momentum, spin, charge and etc. One distinguishes the matter fields (leptons and quarks) and the fields which carry their interactions (photons, gluons and vector bosons).

The quantum field theory is based on two sorts of fundamental physical postulates. The first type of postulates is formulated as a requirement to symmetries of the theory, while the second one determines the quantization procedure. The principle of gauge covariance in the theory of strong and electro-weak interactions enables one to fix almost uniquely the structure of the underlying equations.

The gravitational interactions do not fit well into this scheme. The Einstein equivalence principle of classical general relativity states that results of physical experiments do not depend on the velocity of a locally falling frame of reference. In analogy to field theories, this principle can be formulated as a symmetry requirement, in a form of invariance with respect to local coordinate transformations. The difficulty is that classical gravity theory is essentially non-linear and its quantum version is not known yet. The gravitational field in this book will be always considered classical.

To set the stage for further analysis we give an outlook of the mathematical structure of the field theory including general relativity.

## 1.2 Riemannian Manifolds

We begin with definitions from Riemannian geometry which constitutes mathematical foundation of the general relativity theory.

A real $n$-dimensional manifold $\mathscr{M}$ is a space which looks around each point like a real plane, $\mathbb{R}^n$. More precisely, $\mathscr{M}$ can be covered by (open) subsets $\mathscr{M}_i$ such that for each $i$ there is an injection $f_i : \mathscr{M}_i \to \mathbb{R}^n$ and, if $\mathscr{M}_j$ intersects $\mathscr{M}_k$

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the map $f_j f_k^{-1}$ is smooth. This definition is just a different way to say that one may introduce a coordinate system near each point of the manifold, and that if two coordinate systems overlap, different coordinates of the same point are related by a smooth coordinate transformation.

Clearly, existence of a coordinate system is absolutely necessary for field theory. Since field theory equations contain derivatives with respect to coordinates, one also has to assume that these derivatives are well defined (almost) everywhere, so that the smoothness condition is also understood.

One can define *tensors* as functions on $\mathscr{M}$ having $k$ lower (covariant) and $l$ upper (contravariant) indices and transforming according to the following law:

$$T'^{\nu_1\dots\nu_l}_{\mu_1\dots\mu_k} = \frac{\partial x^{\rho_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\rho_k}}{\partial x'^{\mu_k}} \frac{\partial x'^{\nu_1}}{\partial x^{\sigma_1}} \cdots \frac{\partial x'^{\nu_l}}{\partial x^{\sigma_l}} T^{\sigma_1\dots\sigma_l}_{\rho_1\dots\rho_k}. \tag{1.1}$$

Tensors with a single index, such as $V^\mu$ or $V_\mu$, are called vectors. Tensors without indices are called scalars. They do not change under coordinate transformations, $\varphi'(x') = \varphi(x)$.

The important notion is the *Riemannian* manifold which is a manifold equipped at each its point with a symmetric non-degenerate tensor $g_{\mu\nu}(x)$. This tensor is called the metric tensor or, simply, the metric. The metric tensor is symmetric and can be used to define the scalar product of two vectors in the given point,

$$(V_1 \cdot V_2)_g(x) \equiv V_1^\mu(x) g_{\mu\nu}(x) V_2^\nu(x).$$

We also define the inverse metric $g^{\sigma\rho}$ by the equation: $g^{\sigma\nu} g_{\nu\mu} = \delta_\mu^\sigma$ where $\delta$ is the Kronecker symbol.

At any given point of $\mathscr{M}$ the metric tensor can be brought by a coordinate transformation to the form $g_{\mu\nu}(x) = \eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is the flat metric, i.e. a purely diagonal matrix with certain number of $-1$'s and $+1$'s on its diagonal. This set of minuses and pluses, $(n_-, n_+)$, is called the signature of the metric. The signature must be constant across the manifold, and $n_- + n_+ = n$ since we suppose that the metric is non-degenerate. The manifolds with the signature $(1, n-1)$ are called the *Lorentzian* manifolds. They play a particularly important role in physics since our space-time is an example of such a manifold. In applications to general relativity the diagonal elements are $-1, +1, +1, +1$. A flat manifold with such signature is called Minkowski space-time. In general, a manifold with the signature $(n_-, n_+)$ where $n_\pm \neq 0$ is called pseudo-Riemannian (or pseudo-Euclidean). Locally these manifolds look like $\mathbb{R}^{n_-,n_+}$ (which is $\mathbb{R}^{n_-+n_+}$ with a constant metric of the signature $(n_-, n_+)$). If the metric is positive definite, $n_- = 0$, the manifold is called Euclidean. In what follows we call Euclidean Riemannian manifolds simply Riemannian manifolds.

On Riemannian manifolds the metric tensor determines an interval between two nearby points $x^\mu$ and $x^\mu + dx^\mu$

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu. \tag{1.2}$$

Another important notion is a transport of a vector $V(x)$ (or a tensor) from a point $x^\mu$ to the nearby point $x^\mu + dx^\mu$. The transport is defined to preserve the scalar product of any two transported vectors,

$$(V_1 \cdot V_2)_g(x) = (\tilde{V}_1 \cdot \tilde{V}_2)_g(x + dx). \tag{1.3}$$

Under this transport the components of the vector change as $\tilde{V}(x + dx) = V(x) + \delta V(x)$. The variation $\delta V(x)$ is proportional to $dx^\mu$ and to the components of $V(x)$ (the latter follows from linearity of (1.3) with respect of each of the arguments). For a contravariant vector one can write $\delta V^\mu(x) = -\Gamma^\mu_{\lambda\nu}(x) V^\lambda(x)\, dx^\nu$, where $\Gamma^\mu_{\lambda\nu}(x)$ is a three-index object which depends on the coordinates. The condition (1.3) alone is not enough to fix the transport uniquely. A distinguished case is the *parallel* transport. By analogy with flat space, it is defined by the requirement that there is a local coordinate system, such the components of any vector under the transport to an infinitely close point do not change. Thus, in the vicinity of each point there are coordinate transformations which null the coefficients $\Gamma^\mu_{\lambda\nu}(x)$. Together with (1.3) this condition yields the so-called Christoffel connection

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \tag{1.4}$$

Only these connections are used in this Chapter. Another way to infer these connections from (1.3) is to require the symmetry $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$. This latter condition is the same as the absence of *torsion*.

The parallel transport can be used now to introduce the covariant derivative for tensorial objects. The covariant derivative of a vector $\nabla_\mu V(x)$ is defined through the difference between the value of the vector $V(x + dx)$ at the point $x^\mu + dx^\mu$ and the result of the parallel transport to this point of the vector $V(x)$,

$$\nabla_\mu V(x)\, dx^\mu \equiv V(x + dx) - \tilde{V}(x + dx). \tag{1.5}$$

For contravariant vectors this immediately yields

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda. \tag{1.6}$$

By the construction, the two-index object $\nabla_\mu V^\nu$ is a tensor.

The operation of the parallel transport can be extended to an arbitrary rank tensor and enables one to define the covariant derivative

$$\begin{aligned} \nabla_\mu T^{\sigma_1 \dots \sigma_l}_{\rho_1 \dots \rho_k} &= \partial_\mu T^{\sigma_1 \dots \sigma_l}_{\rho_1 \dots \rho_k} - \Gamma^{\sigma_1}_{\mu\nu} T^{\nu\sigma_2 \dots \sigma_l}_{\rho_1 \dots \rho_k} - \Gamma^{\sigma_2}_{\mu\nu} T^{\sigma_1 \nu \dots \sigma_l}_{\rho_1 \dots \rho_k} - \cdots \\ &\quad + \Gamma^\nu_{\mu\rho_1} T^{\sigma_1 \dots \sigma_l}_{\nu\rho_2 \dots \rho_k} + \Gamma^\nu_{\mu\rho_2} T^{\sigma_1 \dots \sigma_l}_{\rho_1 \nu \dots \rho_k} + \cdots \end{aligned} \tag{1.7}$$

such that the object $\nabla_\mu T^{\sigma_1 \dots \sigma_l}_{\rho_1 \dots \rho_k}$ is a tensor. For scalars the covariant derivative coincides with the partial derivative. It can be checked with the help of (1.7) that

$$\nabla_\mu g_{\nu\rho} \equiv 0. \tag{1.8}$$

This identity is a consequence of the fact that parallel transport preserves the scalar product, see (1.3). From now on we may move indices up and down, $V^\mu = g^{\mu\nu} V_\nu$, $V_\nu = g_{\mu\nu} V^\mu$. Due to (1.8), this operation commutes with the covariant derivative.

One can always introduce a coordinate system such that first partial derivatives of $g_{\mu\nu}$ vanish at a given point. Equation (1.4) implies then that $\Gamma^\rho_{\mu\nu} = 0$ at this point. We shall call such coordinate systems locally inertial (by borrowing this notion from relativity theory). Of course, in general second derivatives of the metric and first derivatives of the Christoffel connection are not zero even in a locally inertial frame.

We say that $\mathscr{M}$ is flat if at each point (i.e. globally) the metric can be brought the to the flat form $g_{\mu\nu} = \eta_{\mu\nu}$ by a coordinate transformation. Since $\Gamma^{\rho}_{\mu\nu} = 0$, the covariant derivatives commute, and since the covariant derivatives are tensorial objects, they commute in any coordinate system. One may introduce an object which characterizes how far a manifold $\mathscr{M}$ differs from the flat one, in other words, how curved it is. This object is called the Riemann tensor and is defined through the equation

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V_\rho = -V_\sigma R^\sigma{}_{\rho\mu\nu}, \tag{1.9}$$

where $V$ is an arbitrary vector. Explicitly,

$$R^\lambda{}_{\mu\nu\kappa} = \partial_\nu \Gamma^\lambda_{\mu\kappa} - \partial_\kappa \Gamma^\lambda_{\mu\nu} + \Gamma^\eta_{\mu\kappa} \Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\kappa\eta}. \tag{1.10}$$

It is convenient to consider also the Riemann tensor with all indices down, $R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^\lambda{}_{\nu\rho\sigma}$. From the definition it is clear that this tensor has the following symmetry properties:

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} = R_{\sigma\rho\mu\nu}. \tag{1.11}$$

The cyclicity property,

$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0, \tag{1.12}$$

is a little bit harder, but also follows the same way. We also note the Bianchi identity:

$$R_{\mu\nu\lambda\rho;\sigma} + R_{\mu\nu\sigma\lambda;\rho} + R_{\mu\nu\rho\sigma;\lambda} = 0, \tag{1.13}$$

where we used the semicolon to denote covariant derivatives of a tensor,

$$T^{\cdots}_{\cdots;\nu_1\ldots\nu_k} := \nabla_{\nu_k} \ldots \nabla_{\nu_1} T^{\cdots}_{\cdots}. \tag{1.14}$$

The Bianchi identity (1.13) follows from the commutation (Jacobi) identity

$$[\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] = 0.$$

We stress, that the Riemann tensor is *not* covariantly constant in general. If $\nabla_\mu R_{\nu\rho\sigma\lambda} = 0$ for all $\nu$, $\mu$, $\rho$, $\sigma$ and $\lambda$ the manifold $\mathscr{M}$ looks locally as a product of spheres with the standard metric (cf. Exercise 1.3).

One can construct two other important objects by contracting indices in the Riemann tensor. These are the Ricci tensor,

$$R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}, \tag{1.15}$$

and the scalar curvature

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{1.16}$$

Obviously, the Ricci tensor is symmetric, $R_{\mu\nu} = R_{\nu\mu}$.

If the manifold $\mathscr{M}$ is *orientable*, one can introduce another important object, namely the $n$ rank Levi-Civita tensor $\varepsilon^{\mu_1\mu_2\cdots\mu_n}$. We fix some numbering of the coordinates $x^1, \ldots, x^n$ and define an $n$-index Levi-Civita *symbol* $\tilde{\varepsilon}$ by requiring (i) that $\tilde{\varepsilon}^{12\ldots n} = 1$, and (ii) that $\tilde{\varepsilon}^{\mu_1\mu_2\ldots\mu_n}$ is totally antisymmetric. This object is globally well defined if and only if $\mathscr{M}$ is orientable. The Levi-Civita *tensor* reads then $\varepsilon^{\mu_1\mu_2\ldots\mu_n} = g^{-1/2}\tilde{\varepsilon}^{\mu_1\mu_2\ldots\mu_n}$, where $g = \det g_{\mu\nu}$ (see Exercise 1.2). This object is parity-odd, i.e. it changes the sign if one reverses the orientation on $\mathscr{M}$. It is convenient to use the Levi-Civita symbol for calculation of determinants. For example,

$$\det g_{\mu\nu} = \frac{1}{n!} g_{\mu_1\nu_1} \ldots g_{\mu_n\nu_n} \tilde{\varepsilon}^{\mu_1\ldots\mu_n} \tilde{\varepsilon}^{\nu_1\ldots\nu_n}. \tag{1.17}$$

It is also easy to derive that $\varepsilon_{\mu_1\mu_2\ldots\mu_n} = g^{1/2}\tilde{\varepsilon}^{\mu_1\mu_2\ldots\mu_n}$, and that the Jacobian of a general coordinate transformation can be represented as follows:

$$\det\left(\frac{\partial x^{\mu'}}{\partial x^{\nu}}\right) = \frac{1}{n!}\frac{\partial x^{\mu'_1}}{\partial x^{\nu_1}} \ldots \frac{\partial x^{\mu'_n}}{\partial x^{\nu_n}} \varepsilon^{\nu_1\ldots\nu_n} \varepsilon_{\mu'_1\ldots\mu'_n}. \tag{1.18}$$

For any antisymmetric tensor $F_{\mu_1\ldots\mu_p}$ of rank $p$ we can define a Hodge dual tensor

$$*F_{\mu_1\ldots\mu_{n-p}} = \frac{1}{p!(n-p)!} \varepsilon_{\mu_1\ldots\mu_{n-p}}{}^{\mu_{n-p+1}\ldots\mu_n} F_{\mu_{n-p+1}\ldots\mu_n}, \tag{1.19}$$

which is an antisymmetric tensor of rank $n - p$. Repeating the Hodge star operation twice one gets back the original tensor, perhaps up to a sign, $**F = (-1)^{p(n-p)}F$.

## 1.3 Gravity Action and Dynamical Equations

An important notion in classical and quantum theories is the action functional or simply the action. Vanishing of the first variation of the action determines equations of motion of the corresponding model. Thus, the action may serve as a definition of the model. Symmetry principles are formulated as a requirement that the action preserves its form under certain transformations of its arguments (fields). We give several examples starting from the gravity theory.

**The Einstein Theory** can be determined by the so-called Einstein-Hilbert action

$$I_{EH}[g] = \frac{1}{16\pi G_N} \int_{\mathscr{M}} d^n x \sqrt{-g}(R - 2\Lambda) \tag{1.20}$$

which is defined on Lorentzian manifolds. Here $G_N$ is the Newton constant, $\Lambda$ is the cosmological constant. Note that $d^n x \sqrt{-g}$ is the invariant measure on the manifold, and (1.20) is invariant under the general coordinate transformations. The total gravity action is $I[\varphi, g] = I_{EH}[g] + I_m[\varphi, g]$ where $I_m[\varphi, g]$ is a contribution from matter fields $\varphi$. Examples of $I_m[\varphi, g]$ for a number of models are given below.

The Einstein equations are determined are the extremum conditions for the total action. Variation of $I_{EH}[g] + I_m[\varphi, g]$ with respect to the metric yields (see formulae (1.104)–(1.107))

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{1.21}$$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta I_m}{\delta g_{\mu\nu}}. \tag{1.22}$$

The "source" on the right hand side of (1.21) is called the stress-energy tensor $T_{\mu\nu}$ of matter fields. In (1.21), the system of units where the speed of light is equal to unity is implied.

In the Riemannian gravity one uses a Euclidean Einstein-Hilbert action with the convention

$$I_{EH}[g] = -\frac{1}{16\pi G_N} \int_{\mathcal{M}} d^n x \sqrt{g}(R - 2\Lambda). \tag{1.23}$$

The solutions with $T_{\mu\nu} = 0$ are called the vacuum solutions. If a Riemannian manifold solves the vacuum Einstein equations for some $\Lambda$ it is called the Einstein manifold. The Ricci tensor on a Einstein manifold is covariantly constant, $R_{\mu\nu;\rho} = 0$.

**Particles in a Gravitational Field** Consider a pair of points $(A, B)$ on a Lorentzian $\mathcal{M}$ connected by a path $x = x(\tau)$ (where $\tau$ is a real parameter), so that $x(0) = A$ and $x(1) = B$. One can define an interval between $A$ and $B$ along this particular path by the integral

$$D(A, B) = \int_A^B ds = \int_0^1 d\tau \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \tag{1.24}$$

and use this functional as an action for point particles. If $A$ and $B$ are not too far away from each other there is a unique path, called the geodesic, which minimizes $D(A, B)$. If $x(\tau)$ is a geodesic, it satisfies the geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0. \tag{1.25}$$

Equations (1.25) hold when $\tau$ is chosen to be an affine parameter (i.e. $d\tau$ is proportional to the interval $\sqrt{|ds^2|}$). On the Lorentzian manifolds the geodesic lines may have null tangent vector $dx^\mu/d\tau$. They are called null geodesics and describe trajectories of light rays.

It follows from (1.5) and (1.25) that the tangent vector $dx^\mu/d\tau$ is parallel transported along the geodesics.

**Topological Theories** The Levi-Civita tensor can be used to construct topological actions on Riemannian manifolds $\mathcal{M}$. These are the actions which do not depend on the metric and are determined by topological properties of $\mathcal{M}$ only. Consider first a rank $p$ antisymmetric tensor field $A^{(p)}_{\mu_1 \ldots \mu_p}$ and construct a corresponding "field strength"

$$F^{(p+1)}_{\mu_1 \ldots \mu_{p+1}} = \nabla_{[\mu_1} A^{(p)}_{\mu_2 \ldots \mu_{p+1}]}, \tag{1.26}$$

where the square brackets denote antisymmetrization of the enclosed indices. Our normalization conventions include $1/(p+1)!$ so that the antisymmetrization of an already antisymmetric tensor is an identity operation. For example, $F_{[\mu\nu]} =$

$\frac{1}{2}(F_{\mu\nu} - F_{\nu\mu})$. Since the Christoffel symbol is symmetric in the lower indices, it does not contribute to (1.26), and one can replace covariant derivatives by the ordinary ones,

$$F^{(p+1)}_{\mu_1\dots\mu_{p+1}} = \partial_{[\mu_1} A^{(p)}_{\mu_2\dots\mu_{p+1}]}. \tag{1.27}$$

This means that $F^{(p+1)}$ actually does not depend on the metric. Let us choose natural numbers $q_1, q_2, \dots, q_k$ such that $q_1 + q_2 + \cdots + q_k = n$ and consider a *Chern-Simons type* action

$$\begin{aligned} I_{CS} &= \int_{\mathscr{M}} d^n x\, g^{1/2} \varepsilon^{\mu_1\dots\mu_{q_1}\nu_1\dots\nu_{q_2}\dots\rho_1\dots\rho_{q_k}} A^{(q_1)}_{\mu_1\dots\mu_{q_1}} F^{(q_2)}_{\nu_1\dots\nu_{q_2}} \dots F^{(q_k)}_{\rho_1\dots\rho_{q_k}} \\ &= \int_{\mathscr{M}} d^n x\, \tilde{\varepsilon}^{\mu_1\dots\mu_{q_1}\nu_1\dots\nu_{q_2}\dots\rho_1\dots\rho_{q_k}} A^{(q_1)}_{\mu_1\dots\mu_{q_1}} F^{(q_2)}_{\nu_1\dots\nu_{q_2}} \dots F^{(q_k)}_{\rho_1\dots\rho_{q_k}}. \end{aligned} \tag{1.28}$$

From the representation on the first line of (1.28) it is obvious that $I_{CS}$ is diffeomorphism invariant. The second line of (1.28) tells us that $I_{CS}$ is topological, i.e. it does not depend on the metric.

**The Euler Characteristic** For a closed Riemannian manifold $\mathscr{M}$ of even dimension $n = 2p$ one can introduce a functional

$$\begin{aligned} &\big(2^{2(p+1)} \pi^p p!\big) \chi_p[\mathscr{M}] \\ &\quad = \int_{\mathscr{M}} \sqrt{g}\, d^{2p} x\, \varepsilon_{\mu_1\mu_2\dots\mu_{2p-1}\mu_{2p}} \varepsilon^{\nu_1\nu_2\dots\nu_{2p-1}\nu_{2p}} R^{\mu_1\mu_2}{}_{\nu_1\nu_2} \dots R^{\mu_{2p-1}\mu_{2p}}{}_{\nu_{2p-1}\nu_{2p}}. \end{aligned} \tag{1.29}$$

One can show that $\chi_p[\mathscr{M}]$ is a topological invariant which takes an integer value. It is called the Euler number. For example, for a 2-sphere $S^2$ the Euler number equals 2. Definition (1.29) can be extended to manifolds with a boundary.

## 1.4 Physical Examples of Manifolds

**De Sitter Space** We begin with a simple example in four dimensions. It is the de Sitter solution to the vacuum Einstein equations (1.21) with a positive cosmological constant, $\Lambda > 0$,

$$ds^2 = -dt^2 + a^2 \cosh^2\left(\frac{t}{a}\right) d\Omega_3^2. \tag{1.30}$$

Here $a \equiv \sqrt{\frac{3}{\Lambda}}$ and

$$d\Omega_3^2 = d\rho^2 + \sin^2\rho\,(\sin^2\theta\, d\varphi^2 + d\theta^2)$$

is the metric on a three-dimensional hypersphere $S^3$ with unit radius.

Metric (1.30) can be used in applications to cosmological models. It describes a universe which first contracts till the moment $t = 0$ and then expands.

The element (1.30) is the metric of a hypersurface embedded in a flat Minkowski space one dimension higher

$$-(X_0)^2 + (X_1)^2 + (X_2)^2 + (X_3)^2 + (X_4)^2 = a^2. \tag{1.31}$$

This hypersurface is called the de Sitter space and is sometime denoted as $dS_4$. The parametrization which corresponds to (1.30) is

$$X_0 = a \sinh\left(\frac{t}{a}\right), \qquad X_4 = a \cosh\left(\frac{t}{a}\right) \cos\rho,$$

$$X_1 = a \cosh\left(\frac{t}{a}\right) \sin\rho \sin\theta \sin\varphi, \qquad X_2 = a \cosh\left(\frac{t}{a}\right) \sin\rho \sin\theta \cos\varphi,$$

$$X_3 = a \cosh\left(\frac{t}{a}\right) \sin\rho \cos\theta.$$

The de Sitter space is maximally symmetric and has a positive curvature. As follows from (1.31), its group of isometries is $SO(1,4)$.

The de Sitter space is an example of a Lorentzian manifold. In the Euclidean Einstein theory an analog of the de Sitter space is hypersphere $S^4$. It is a maximally symmetric space with positive curvature and the isometry group $SO(5)$. One can get from the de Sitter metric the metric on $S^4$ by assuming imaginary time $t = i\eta$ in (1.30).

Another important space which is called anti-de Sitter space (AdS) is obtained under the following changes in (1.31): $X_1^2 \to -X_1^2$ and $a^2 \to -a^2$. It has the isometry group $SO(2,3)$.

**Black Hole Solutions** One of the most interesting solutions of the Einstein equations are those which describe black holes, i.e. regions of space-time where the gravitational field is so strong that nothing including light signals can escape these regions and reach an external observer. The imaginary surface which separates the "visible" and "invisible" (from the point of view an external observer) regions of space is generated by null geodesics and is called the horizon of the black hole.

We give a simple example of a static black hole which is the Schwarzschild solution to the vacuum Einstein equations with zero cosmological constant. In the region outside the horizon the line element of this solution is

$$ds^2 = -\left(1 - \frac{r_H}{r}\right) dt^2 + \left(1 - \frac{r_H}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{1.32}$$

where $d\Omega_2$ is the line element on the unit two-sphere

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta \, d\varphi^2. \tag{1.33}$$

Here it is assumed that $r > r_H$. $r_H$ is a parameter, $r = r_H$ corresponds the position of the horizon. In terms of the mass $M$ of the black hole $r_H = 2MG$, and, by going to physical units, one can easily find [251] that $r_H = (M/M_\odot) \times 3$ km where $M_\odot$ is the mass of the Sun. In general, (1.32) describes the gravitational field of a spherical body of the mass $M$ and some radius $R$ (in this case $r \geq R > r_H$).

The Schwarzschild coordinates (1.32) are singular at $r = r_H$. It is a pure coordinate singularity because geodesic lines of particles falling in the black hole can be smoothly continued inside the horizon. The latter fact means that Schwarzschild coordinates are defined on a chart which covers only a part of the entire black hole geometry. To see this consider, for example, an ingoing light ray whose trajectory is given by the simple equation $t + r^* = const$, where

$$r^* = r + r_H \ln\left|\frac{r - r_H}{r_H}\right| \tag{1.34}$$

is the Regge-Wheeler coordinate. Let us introduce the new coordinate $v = t + r^*$, then (1.32) takes the form

$$\begin{aligned} ds^2 &= \left(1 - \frac{r_H}{r}\right)(-dt^2 + dr^{*2}) + r^2\, d\Omega_2^2 \\ &= -\left(1 - \frac{r_H}{r}\right) dv^2 + 2dr\, dv + r^2\, d\Omega_2^2. \end{aligned} \tag{1.35}$$

These coordinates are called the ingoing Eddington-Finkelstein coordinates because trajectories of ingoing null geodesics here are simply $v = const$. In these coordinates $\det g_{\mu\nu}$ is nonsingular and, therefore, the metric is invertible and nonsingular at $r = r_H$. Thus, one can extend the domain of definition of $r$ to non-negative values, $0 < r < \infty$. The point $r = 0$ is a real singularity where the curvature tensor is blowing up. Indeed, one can check that $R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} \sim \frac{r_H^2}{r^6}$. More details about local and global geometry of black holes can be found in [238, 251].

**Black Hole Instantons** If in (1.32) we make the time coordinate purely imaginary, $t = i\tau$, we obtain a metric on a Riemannian manifold

$$ds^2 = \left(1 - \frac{r_H}{r}\right) d\tau^2 + \left(1 - \frac{r_H}{r}\right)^{-1} dr^2 + r^2\, d\Omega_2^2. \tag{1.36}$$

The surface $r = r_H$, called the Euclidean horizon, differs from horizon in the Lorentzian geometry. Let us introduce the coordinate

$$\rho = \int_{r_H}^{r} \frac{dr'}{\sqrt{1 - \frac{r_H}{r'}}}$$

such that in the position of the Euclidean horizon becomes $\rho = 0$. Near $\rho = 0$ the metric (1.36) reads

$$ds^2 = \kappa^2 \rho^2\, d\tau^2 + d\rho^2 + r_H^2\, d\Omega_2^2, \tag{1.37}$$

where $\kappa = 1/(2r_H)$. Regularity of the solution at $\rho = 0$ requires that $\tau$ is a cyclic coordinate with the period $2\pi/\kappa$. If this is the case the geometry described by metric (1.36) is smooth everywhere and is a solution to the Euclidean vacuum Einstein equations without cosmological constant. It is called the black hole instanton and is used to describe thermodynamic properties of a black hole.

## 1.5 Fiber Bundles and Matter Fields

In this book we study quantum effects due to matter fields on an external background. Therefore, we first need to describe classical matter fields on Riemannian manifolds. The fields are functions on a manifold which are characterized by some internal structure. For example, they may belong to the space of a representation of a group of internal symmetries (a gauge group). Fields also have a spin structure. The leptons and quarks have spin 1/2 and are described by spinor fields while the gauge bosons have spin 1 and they are vector fields.

To describe these additional structures we need the notion of fiber bundles. Suppose we have some manifold $\mathscr{M}$ which we shall call the base manifold, and some other manifold $\mathscr{F}$ which we shall call the fiber. A fiber bundle over $\mathscr{M}$ with fiber $\mathscr{F}$ is a manifold which locally looks as a direct product $\mathscr{M} \times \mathscr{F}$. We already know that $\mathscr{M}$ can be covered by a set of local coordinate neighborhoods $\mathscr{M}_j$. Let in each neighborhood the bundle $\mathscr{E}$ be the product manifold $\mathscr{M}_j \times \mathscr{F}$. The global topology of $\mathscr{E}$ is defined by the set of transition functions $\Phi_{ij}$ which tell how the fibers match up in the overlaps $\mathscr{M}_i \cap \mathscr{M}_j$. These functions are maps $\Phi_{ij} : \mathscr{F}|_{\mathscr{M}_i} \to \mathscr{F}|_{\mathscr{M}_i}$ in $\mathscr{M}_i \cap \mathscr{M}_j$.

We require that the transition functions belong to a group $G$ of transformations of the fiber space $\mathscr{F}$. $G$ is called the structure group of the fiber bundle.

Our discussion in this section is based upon [99], where the reader can find more details and formal definitions.

If the fiber bundle is a direct product also globally,

$$\mathscr{E} = \mathscr{M} \times \mathscr{F}, \tag{1.38}$$

it is called trivial fiber bundle.

A section of $\mathscr{E}$ is a rule which takes a point $\varphi(x)$ on each fiber $\mathscr{F}$ for each point of the base manifold $\mathscr{M}$. A local section is defined only over a subset of $\mathscr{M}$. Global sections are defined over the whole $\mathscr{M}$. The existence of global sections depends on the topology of $\mathscr{E}$. There are fiber bundles which admit no global sections. $\varphi(x)$ is what we call a field on $\mathscr{M}$ in the physical context.

In most of the physical applications the field $\varphi(x)$ takes values in a linear space, $\mathscr{F} = \mathbb{R}^k$. Even if the fields themselves do not belong to a linear space (as in the case of gravity or non-linear sigma models), their fluctuations do. Therefore, in what follows we restrict ourselves to the case $\mathscr{F} = \mathbb{R}^k$. Then transition functions $\Phi_{ij}$ belong to $GL(k, \mathbb{R})$. Such bundles are called vector bundles. $k$ is called the bundle dimension. One can replace $\mathbb{R}^k$ by $\mathbb{C}^k$ to obtain a complex vector bundle.

Consider two vector bundles $\mathscr{E}_1$ and $\mathscr{E}_2$ with the same base manifold $\mathscr{M}$. One can define a tensor product bundle $\mathscr{E}_1 \otimes \mathscr{E}_2$ by taking the tensor product of the fibers $\mathscr{F}_1$ and $\mathscr{F}_2$ at each point of $\mathscr{M}$. Similarly one can define the Whitney sum $\mathscr{E}_1 \oplus \mathscr{E}_2$ by taking the direct sum of the fibers.

For each vector space $\mathscr{F}$ one can define the dual vector space $\mathscr{F}^*$ as a space of linear maps from $\mathscr{F}$ to $\mathbb{R}$. Therefore, one can also define the dual vector bundle $\mathscr{E}^*$ to $\mathscr{E}$ as a bundle whose fibers are $\mathscr{F}^*$ pointwise on $\mathscr{M}$. The fiber $\mathscr{F}$ can be equipped with an inner product (called the fiber metric). In such a case, one can define an inner product of sections. This construction is particularly important for quantization. A fiber metric defines a linear isomorphism between $\mathscr{E}$ and $\mathscr{E}^*$. The examples of inner products will be given below.

An important example of a bundle is the so-called principal bundle $P$. The fiber of this bundle is a Lie group $G$ (which is a manifold). Transition functions of $P$ belong to $G$ and act by left multiplications. Principal bundles can be used to define gauge transformations with the gauge group $G$.

The transition functions $\Phi_{ij}$ have been introduced above as maps between fibers on the overlaps of two neighborhoods. One can also view these functions as local changes of local bases in the bundle or as local gauge transformations. Local sections of vector bundles change covariantly under these transformations. That means that if $\varphi(x)$ is a section, and if $\mathbf{g}(x) \in G$ (where $G$ is the structure group of the bundle), then $\varphi(x)$ is mapped to $\mathbf{g}(x)\varphi(x)$. Obviously, $\partial_\mu\varphi(x)$ is not a covariant object. To make derivatives covariant one has to introduce a connection $\omega_\mu$, so that the covariant derivative

$$\nabla_\mu\varphi(x) = \partial_\mu\varphi(x) + \omega_\mu(x)\varphi(x) \tag{1.39}$$

is indeed covariant provided we postulate the following transformation rule for $\omega_\mu$:

$$\omega_\mu \to \mathbf{g}\partial_\mu\mathbf{g}^{-1} + \mathbf{g}\omega_\mu\mathbf{g}^{-1}. \tag{1.40}$$

In a local basis $\omega_\mu$ is just a matrix-valued function. One introduces also a field strength

$$\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + \omega_\mu\omega_\nu - \omega_\nu\omega_\mu, \tag{1.41}$$

which is also covariant.

We consider now the so-called tangent and cotangent bundles. First we introduce tangent and cotangent spaces. To define the tangent space $T_x(\mathscr{M})$ to the manifold $\mathscr{M}$ at a point $x$ one takes a function $f(x)$ and expands it in a Taylor series near $x$:

$$f(x+\xi) = f(x) + \xi^\mu\partial_\mu f(x) + \cdots \tag{1.42}$$

Vector fields on $\mathscr{M}$ are then identified with the directional derivatives $\xi^\mu\partial_\mu$ with smoothly varying coefficients $\xi^\mu$. The tangent space $T_x(\mathscr{M})$ is then defined as a vector space spanned by the tangents at $x$ to all curves passing through $x$. The tangent bundle $T(\mathscr{M})$ is a vector bundle whose fibres at a point $x \in \mathscr{M}$ are given by the tangent space $T_x(\mathscr{M})$. The cotangent space is defined as the dual $T_x^*(\mathscr{M})$ to the vector space $T_x(\mathscr{M})$. The cotangent bundle $T^*(\mathscr{M})$ is the dual to $T(\mathscr{M})$. One says that a natural local basis on $T(\mathscr{M})$ is given by partial derivatives $\{\partial_\mu\}$, and that a natural local basis for $T^*(\mathscr{M})$ is given by differential one-forms $\{dx^\mu\}$. The bundle dimension of $T(\mathscr{M})$ and $T^*(\mathscr{M})$ is obviously $n$.

The reader may get an impression that the definitions in the preceding paragraph were too abstract and that they came too fast. In fact, the construction above is presented here just to provide a bridge to the mathematics literature (cf. [99], where we took this material from). For our purpose it will be enough to identify vector fields (which are derivatives according to the definition above) with the coefficient functions $\xi^\mu$, i.e. with contravariant vectors as we defined them in the previous section. Their "duals" are then covariant vector fields. Now, it is easy to figure out what the tangent and cotangent bundles really are.

The Riemannian metric is a natural fiber metric on $T(\mathscr{M})$ and $T^*(\mathscr{M})$. Thus, on a Riemannian manifold, $T(\mathscr{M})$ is always isomorphic to $T^*(\mathscr{M})$, and this isomorphism simply moves the vector indices up and down with the help of the metric $g_{\mu\nu}$. In what follows we do not make much distinction between the tangent and cotangent bundles.

Locally on a Riemannian manifold (with a positive definite metric) one can always introduce an orthonormal basis $e_a^\mu$, $a = 1, \dots, n$ such that

$$e_a^\mu(x) e_b^\nu(x) g_{\mu\nu}(x) = \delta_{ab}. \tag{1.43}$$

We follow the convention that the letters from the beginning of the Latin alphabet are used to enumerate the elements of the basis. They are called "flat" or "tangential" indices as opposed to the vector indices denoted by the Greek letters and called "curved". The basis vector $e_\mu^a$ are called the vielbeins. According to (1.43) the vielbeins can be interpreted as a "square root of the metric".

There is a dual basis $e_\mu^a = \delta^{ab} g_{\mu\nu} e_b^\nu$ such that the following relations hold:

$$e_\mu^a e_b^\mu = \delta_b^a, \qquad e_\mu^a e_a^\nu = \delta_\mu^\nu, \qquad e_\mu^a e_\nu^b \delta_{ab} = g_{\mu\nu}. \tag{1.44}$$

One can move flat indices up and down with the help of the Kronecker symbol $\delta_{ab}$ as we did it with the curved indices with the help of the metric. With the help of the vielbein one can transform flat indices to curved indices and vice versa,

$$v^a = e_\mu^a v^\mu, \qquad v^\mu = e_a^\mu v^a. \tag{1.45}$$

Since there exists a positive definite fiber metric (induced by the Riemannian metric) which must be preserved by the transition functions $\Phi_{ij}$, the structure group of the bundle is reduced to $O(n)$. In particular, the relation between vectors of different basis sets is

$$\tilde{e}_a^\mu(x) = O_a{}^b(x) e_b^\mu(x), \tag{1.46}$$

where $O_a{}^b(x)$ make elements of matrices which belong to $O(n)$.

Let us now introduce a connection. The covariant derivative is defined as usual,

$$\nabla_\mu v^a = \partial_\mu v^a + w_\mu{}^a{}_b v^b. \tag{1.47}$$

We define the covariant derivative for tensorial object with several indices by extending the rule from the previous section. Namely, we put one $w$-connection for each flat index and one Christoffel connection for each curved index. For example,

$$\nabla_\mu T_{b\nu}^a = \partial_\mu T_{b\nu}^a + w_\mu{}^a{}_c T_{b\nu}^c + w_{\mu b}^c T_{c\nu}^a - \Gamma_{\mu\nu}^\rho T_{b\rho}^a. \tag{1.48}$$

There could be, of course, various choices for the connection. However, we would like to have a connection which is consistent with all other structures we have already defined on the manifold. In particular, we require that covariant differentiation commutes with contractions of flat indices. This implies

$$0 = \nabla_\mu \delta^{ab} = w_\mu{}^{ab} + w_\mu{}^{ba}, \tag{1.49}$$

i.e. the connection is antisymmetric. Consequently, $\omega_\mu$ belongs to the Lie algebra $so(n)$ of the structure group $O(n)$. Next we require that the vielbein is covariantly constant,

$$\nabla_\mu e^a_\nu = 0. \tag{1.50}$$

This condition can be solved for $\omega_\mu$:

$$w_\mu{}^{ab} = e^{\nu b}\Gamma^\rho_{\mu\nu} e^a_\rho - e^{\nu b}\partial_\mu e^a_\nu. \tag{1.51}$$

This connection is called the Levi-Civita connection. The field strength for this connection is defined by the Riemann curvature tensor

$$\partial_\mu w_\nu{}^{ab} - \partial_\nu w_\nu{}^{ab} + w_\mu{}^a{}_c w_\nu{}^{cb} - w_\mu{}^a{}_c w_\mu{}^{cb} = R^{ab}{}_{\mu\nu}, \tag{1.52}$$

see Exercise 1.7.

Now we can describe all tensor fields in the vector bundle language. But what about the spinors? We need the so-called spin bundles. The structure group in this case is the spin group $Spin(n)$ which can be introduced as follows. Consider the algebra of the Dirac gamma-matrices, $\gamma^a$, $a = 1, \ldots, n$. These are Hermitian traceless $2^{[n/2]}$ by $2^{[n/2]}$ matrices, which obey the so-called Clifford anticommutation relation

$$\{\gamma^a, \gamma^b\} = \gamma^a\gamma^b + \gamma^b\gamma^a = 2\delta^{ab}\mathbb{I}, \tag{1.53}$$

where $\mathbb{I}$ is a unit $2^{[n/2]}$ by $2^{[n/2]}$ matrix. In what follows in relations with the gamma-matrices we shall not write $\mathbb{I}$ explicitly. There are infinitely many sets of $\gamma$'s which satisfy (1.53). The matrices of one set are linear combinations of the matrices of the other set, $\tilde{\gamma}^a = O^a_b\gamma^b$. It follows from (1.53) that matrices $O^a_b$ belong to $O(n)$. The Clifford algebra is also invariant with respect to unitary transformations $S\gamma^a S^+$, $S^+ = S^{-1}$. The two sorts of transformations can be related,

$$S\gamma^a S^+ = O^a_b\gamma^b. \tag{1.54}$$

Equation (1.54) can be used to define the elements $S$ of the spin group $Spin(n)$ if matrices $O^a_b$ belong to the group $SO(n)$ (have unit determinant). Because each element of $SO(n)$ corresponds to two elements, $S$ and $-S$, one says that $Spin(n)$ is a double covering of $SO(n)$. A consequence of this fact, known from text books on quantum mechanics, is that a spinor changes its sign under a rigid rotation by the angle $2\pi$.

Equation (1.54) is used to relate the structure groups $Spin(n)$ and $SO(n)$. A spinor field, $\psi(x)$, on a Riemannian manifold $\mathscr{M}$ belongs to representation of the spin group. Under a change of the basis (1.46) the spinor transforms as $\psi'(x) = S(x)\psi(x)$ where matrices $S$ satisfy (1.54). With respect to coordinate transformations $\psi(x)$ changes as a set of scalars. In what follows we assume that the base manifold $\mathscr{M}$ admits a well-defined spinor structure although it is not so in general because of topological obstructions.

We have to introduce an action functional for spinors which is invariant with respect to coordinate transformations and the structure group transformations. To this aim we use vielbeins to construct at the each point of $\mathscr{M}$ a local set of gamma-matrices $\gamma^\mu(x) = e^\mu_a(x)\gamma^a$ satisfying the following Clifford relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{1.55}$$

Note that $\gamma^\mu$'s depend on coordinates while $\gamma^a$'s do not.

Next we have to define a suitable connection on the spin bundle. As we have observed above, the components of the connection on the tangent bundle are simply the generators of the rotation group. Therefore, it is natural to suppose that the connection in the spinor bundle $\omega_\mu^{[s]}$ is proportional to the same generators but taken in the spinor representation of corresponding Lie algebra, i.e. $\omega_\mu^{[s]} \sim \omega_\mu{}^{ab}[\gamma_a, \gamma_b]$. The coefficient can be recovered either on some group theoretical grounds, or by demanding that the gamma-matrices commute with the covariant derivative

$$[\nabla_\mu, \gamma^a] = [w_\mu^{[s]}, \gamma^a] + w_\mu{}^a{}_b \gamma^b = 0. \tag{1.56}$$

(Note that if $\gamma^a$ appears on the right from the covariant derivative the index $a$ should also be contracted with an appropriate connection term.) This condition yields the so-called spin-connection

$$w_\mu^{[s]} = \frac{1}{8} w_\mu{}^{ab}[\gamma_a, \gamma_b]. \tag{1.57}$$

The corresponding field strength is again given by the Riemann curvature tensor

$$\partial_\mu w_\nu^{[s]} - \partial_\nu w_\mu^{[s]} + w_\mu^{[s]} w_\nu^{[s]} - w_\nu^{[s]} w_\mu^{[s]} = \frac{1}{4} \gamma^a \gamma^b R_{ab\mu\nu}. \tag{1.58}$$

If $n$ is even one can define a chirality matrix $\gamma_* = i^{n/2}\gamma_1\gamma_2 \dots \gamma_n$. One can check that

$$\gamma_* \gamma^\mu = -\gamma^\mu \gamma_*, \qquad \gamma_\star^2 = 1, \qquad \gamma_\star^\dagger = \gamma_*. \tag{1.59}$$

In four dimensions $\gamma_*$ is usually denoted as $\gamma_5$, and sometimes the same notation is used for other $n$.

The existence of the chirality matrix shows that in even dimensions $n$ the spin group has two representations given by $2^{n/2-1}$ by $2^{n/2-1}$ matrices.

Now, we know how to incorporate arbitrary spin-tensor fields in the vector bundle scheme. Additional gauge indices are not a problem as well. One should simply add to the covariant derivative a term containing corresponding gauge field, see examples in Sect. 1.6. The main advantage of this approach is that one can consider abstract vector bundles and connections on them. The results will be valid for arbitrary spins and gauge groups.

Till now we discussed Riemannian (Euclidean) manifolds. Some comments about the features of the Lorentzian manifolds are in order. In this case the orthonormal basis contains one time-like vector with a negative norm and $n - 1$ space-like vectors of a positive norm. As a consequence, the "flat" metric which is used to move flat indices up and down should be $\eta_{ab} = \text{diag}(-1, 1, \dots, 1)_{ab}$ instead of $\delta_{ab}$. Thus,

$$e^a_\mu e^b_\mu \eta_{ab} = g_{\mu\nu}, \qquad e^a_\mu e^b_\mu g^{\mu\nu} = \eta^{ab}. \tag{1.60}$$

The structure group of the bundle over a Lorentzian manifold is the Lorentz group $O(1, n-1)$ or the group of pseudo-orthogonal transformations which leaves invariant the quadratic form determined by the metric $\eta_{ab}$.

The definition of $\gamma$-matrices should take into account the signature of the space-time as well. The Clifford commutation relation should read

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{1.61}$$

(cf. (1.53)). The $Spin(1, n-1)$ group is defined by the relation analogous to (1.54),

$$S\gamma^a S^{-1} = \Lambda^a{}_b \gamma^b, \tag{1.62}$$

where $\Lambda^a{}_b$ belong to $SO(1, n-1)$. Generators of the $Spin(1, n-1)$ group can be expressed in terms of the gamma-matrices, see Exercise 1.9. In the Lorentzian theory the gamma-matrix $\gamma_{a=0}$ is anti-Hermitean because $(\gamma_{a=0})^2 = -1$, see (1.61). Therefore, matrices $S$ are not unitary. Instead, one can show (see Exercise 1.10) that

$$\beta S^\dagger \beta = S^{-1}, \tag{1.63}$$

where we choose the following convention: $\beta \equiv i\gamma^{a=0} = -i\gamma_{a=0}$.

By keeping this feature in mind one can give the definition of the spin bundles over Lorentzian manifolds and the definition of the structure $Spin(1, n-1)$ group. The spinor field transforms as

$$\psi'(x) = S(x)\psi(x). \tag{1.64}$$

Its conjugated spinor (the so-called Dirac conjugated spinor) is defined as

$$\bar{\psi} = \psi^\dagger \beta \tag{1.65}$$

and transforms as $\bar{\psi}'(x) = \bar{\psi}(x)S^{-1}(x)$. One can move on and use vielbeins to construct at the each point of $\mathscr{M}$ a local set of gamma-matrices $\gamma^\mu(x) = e^\mu_a(x)\gamma^a$ which satisfy the Clifford relation (1.55) where the metric $g^{\mu\nu}$ now has the signature $(1, n-1)$. The definitions of the covariant derivatives, spin-connection (1.57) and field strength (1.58) are left the same.

On even-dimensional manifolds one can still introduce a chirality matrix $\gamma_*$ which satisfies (1.59). The spin group in even dimensions $n$ has two representations (by $2^{n/2-1}$ by $2^{n/2-1}$ matrices) which are related according to (1.63). In four dimensions the corresponding group $Spin(1, 3)$ is isomorphic to $SL(2, C)$ (complex rank 2 matrices with unit determinant).

Our final remark is related to the notion of conjugated spinors. The spin group transformations for a spinor $\psi$ and its complex conjugation $\psi^*$ are different. One can introduce a spinor $\psi^c$ which transforms as $\psi$ and is called *a charge conjugated spinor*,

$$\psi^c = C\bar{\psi}^T. \tag{1.66}$$

The matrix $C$ is determined by the equations

$$C\gamma_\mu^T C^T = -\gamma_\mu, \tag{1.67}$$

where the superscript $T$ is used for transposed matrices. Explicit construction of the matrix $C$ is discussed in Exercise 1.11.

For certain signatures of the space-time one can take a real representation for the gamma-matrices, which is called the Majorana representation. Then the space of real spinors is invariant with respect to the corresponding *Spin* group. Such spinors are called the Majorana spinors.

## 1.6 Examples of Field Models

Having discussed general properties of matter fields we are ready to give several examples of their action functionals. We require that the considered models are gauge invariant and obey the principle of equivalence. The latter means that the functionals are invariant with respect to coordinate transformations and actions of structure Lorentz and $Spin(1, n-1)$ groups.

**Scalar Model** describes a free charged scalar $\varphi$. The example of scalars is the Higgs field which plays the key role in the standard theory of electroweak interactions. The action of the most simple model is

$$I[\varphi, g, A] = -\int d^n x \sqrt{-g}(g^{\mu\nu}(D_\mu\varphi)^* D_\nu\varphi + m^2\varphi^*\varphi), \tag{1.68}$$

where $D_\mu = \partial_\mu + ieA_\mu$. The metric $g_{\mu\nu}$ and the gauge potential $A_\mu$ are considered as external (not dynamical) fields. The functional (1.68) is invariant with respect to the coordinate transformations and local $U(1)$ gauge transformations $\varphi'(x) = e^{ie\lambda(x)}\varphi(x)$, $A'_\mu(x) = A_\mu(x) - \partial_\mu\lambda(x)$. The equation of motion for the field $\varphi$,

$$(D^\mu D_\mu - m^2)\varphi = 0, \tag{1.69}$$

can be obtained by requiring that the first variation of the action (1.68) with respect to $\varphi$ vanishes. According to (1.22), variation of the action with respect to the metric yields the stress-energy tensor

$$T_{\mu\nu} = 2(D_\mu\varphi)^* D_\nu\varphi - g_{\mu\nu}((D_\sigma\varphi)^* D^\sigma\varphi + m^2\varphi^*\varphi). \tag{1.70}$$

Analogously, variation of $A_\mu$ yields the electric current

$$J^\mu = \frac{1}{\sqrt{-g}}\frac{\delta I}{\delta A_\mu}, \tag{1.71}$$

$$J^\mu = -ie((D^\mu\varphi)^*\varphi - \varphi^* D^\mu\varphi). \tag{1.72}$$

The coordinate and gauge invariance of the action functional imply the following identities $\nabla_\mu T^{\mu\nu} = 0$, $\nabla_\mu J^\mu = 0$ provided that $\varphi$ obeys equation of motion (1.69).

The operator $(D^2 - m^2)$ in (1.69) is a second order hyperbolic type operator (see the definition in Sect. 3.4). According to the general theory of differential equations, a solution to (1.69) can be uniquely fixed by a set of Cauchy data on a space-like hypersurface $\Sigma$. These data include the value of the field on $\Sigma$ and its first normal derivative. Strictly speaking, this procedure requires that $\Sigma$ is *Cauchy hypersurface*, i.e. any non-space-like curve intersects $\Sigma$ exactly once [156]. A Lorentzian space-time is called *globally hyperbolic* space-time if it possesses a Cauchy hypersurface [156]. In what follows we consider models in globally hyperbolic spacetimes.

**The Dirac Fields** $\psi$ describe particles with spin $1/2$, such as electrons, muons and quarks. The action of the model and the equations of motion are

$$I[\psi, g, A] = -\frac{1}{2}\int d^n x \sqrt{-g}\bar{\psi}(\gamma^\mu D_\mu + m)\psi + c.c., \tag{1.73}$$

$$(\gamma^{\mu} D_{\mu} + m)\psi = 0. \tag{1.74}$$

The covariant derivatives are defined as $D_\mu = \nabla_\mu + ieA_\mu$, where $\nabla_\mu = \partial_\mu + w^{[s]}_\mu$ is the spinor covariant derivative. The functional (1.73) possesses $U(1)$ gauge invariance. It is also invariant with respect to coordinate and $Spin(1,3)$ transformations (1.64).

One assumes that the spinor field has an odd Grassmann parity, i.e. that components of the spinor field anticommute. The parity does not play any role in the Dirac equation (1.74) since it is linear. However, considering spinors as commuting variables in the action (1.73) may lead to some inconsistencies already at the classical level, see Exercise 1.12.

**Vector Fields** Charged and neutral vector bosons are observed in experiments. Together with the photon they are responsible for electroweak forces. The classical action for a massive neutral vector field $A_\mu$ is

$$I[A,g] = -\frac{1}{4}\int d^n x\,\sqrt{-g}(F_{\mu\nu}F^{\mu\nu} + 2M^2 A_\mu A^\mu), \tag{1.75}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the Maxwell field strength tensor. Variation of (1.75) results in equation of motion

$$\nabla_\nu F^{\nu\mu} - M^2 A^\mu = 0. \tag{1.76}$$

If $M = 0$ the functional (1.75) is the Maxwell action for photons in an external gravitational field.

**Non-Abelian Gauge Fields** We discuss the theory of non-Abelian gauge fields taking as an example the gauge group $SU(N)$. The fundamental representation of this group is given by unitary $N \times N$ matrices with the unit determinant. The non-Abelian gauge theory with the group $SU(3)$ describes gluons which mediate interactions between quarks. The corresponding quantum theory is called quantum chromodynamics, or QCD. The action of the gauge fields is the Yang-Mills action

$$I[B,g] = \frac{1}{2}\int d^n x\,\sqrt{-g}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}. \tag{1.77}$$

The strength tensor is $F_{\mu\nu} = [D_\mu, D_\nu]$, where $D_\mu = \nabla_\mu + B_\mu$ and the following rule is applied: $[\partial_\mu, f] = (\partial_\mu f) + f\partial_\mu - f\partial_\mu = (\partial_\mu f)$. The gauge fields $B_\mu$ are anti-Hermitian matrices which belong to the fundamental representation of the Lie algebra $su(N)$ of the $SU(N)$ group. The functional (1.77) is invariant under the gauge transformations $B'_\mu = UB_\mu U^{-1} + U\nabla_\mu U^{-1}$, where $U$ is an element of $SU(N)$.

The Yang-Mills equations which determine the extrema of functional (1.77) are

$$[D_\mu, F^{\mu\nu}] = 0. \tag{1.78}$$

Let $T_a$ be a basis of $su(N)$. Then $[T_a, T_b] = f_{abc}T_c$. It can be shown that $f_{abc}$ is a totally anti-symmetric real tensor. One can use normalization, $\mathrm{tr}\,T_a T_b = -\frac{1}{2}\delta_{ab}$, and consider decomposition $B_\mu = B^a_\mu T_a$, where $B^a_\mu$ are real vector fields, $a = 1, \dots, N^2 - 1$.

Let $B_\mu$ be a solution to (1.78). One can consider a small perturbation $A_\mu$ near $B_\mu$. The perturbations obey linearized equations which follow from (1.78)

$$[D^\nu(B), G_{\nu\mu}] + [A^\nu, F_{\nu\mu}(B)] = 0, \tag{1.79}$$

$$G_{\mu\nu} = [D_\mu(B), A_\nu] - [D_\nu(B), A_\mu], \tag{1.80}$$

where the covariant derivatives $D_\mu(B)$ and the strength tensor $F_{\mu\nu}(B)$ are determined in terms of the background field $B_\mu$.

## 1.7 Isometries

An important property of a manifold is a group of its isometries. Isometries are transformations of coordinates which do not change the form of the metric. Transformations of the metric under infinitesimal diffeomorphisms $(x')^\mu = x^\mu - \xi^\mu$ follow from (1.1) and are determined in terms of the Lie derivative

$$\mathscr{L}_\xi g_{\mu\nu} \equiv g_{\mu\nu}(x') - g_{\mu\nu}(x) = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{1.81}$$

Thus, the isometries exist if for a given metric there are vector fields $\xi^\mu(x)$ which are solutions to the equations

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{1.82}$$

Equations (1.82) are called the Killing equations. The solutions to (1.82) are called the Killing vectors.

One can also define the Lie derivatives of matter fields as variation of their form under diffeomorphisms, $\mathscr{L}_\xi \phi(x) = \phi(x') - \phi(x)$. It is easy to see that, e.g., the Lie derivatives of scalar and vector fields are

$$\mathscr{L}_\xi \varphi = \xi^\mu \partial_\mu \varphi, \qquad \mathscr{L}_\xi A^\mu = \xi^\nu \nabla_\nu A^\mu - A^\nu \nabla_\nu \xi^\mu. \tag{1.83}$$

A field configuration $\phi$ preserves its form under isometry transformation generated by a Killing vector field $\xi$ if the corresponding Lie derivative vanishes $\mathscr{L}_\xi \phi = 0$.

One can prove that a manifold of the dimension $n$ may have no more than $n(n+1)/2$ Killing vectors. For example, the hyperplane $\mathbb{R}^n$ has $n(n+1)/2$ isometries which are rotations and translations. Manifolds having maximal number of Killing vectors are called maximally symmetric. The Riemann tensor on such manifolds reads

$$R_{\mu\nu\rho\sigma} = C(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}), \tag{1.84}$$

where $C = R/(n(n-1))$, and $R$ is the scalar curvature. The scalar curvature of any maximally symmetric space is constant.

On a Lorentzian manifold, there is a special class of isometries which plays an important role in applications and, in particular, for interpreting properties of field excitations. These are isometries with respect to a time-like Killing vector field. They are global time translations. If such a Killing vector exists, the components of the metric tensor do not depend on time in a suitable coordinate system. Then the manifold $\mathscr{M}$ is called stationary.

On a maximally symmetric manifold $\mathscr{M}$ one can define the so-called Killing spinors. These are solutions to the equations

$$D_\mu \epsilon = \left[\nabla_\mu + \frac{ia}{2}\gamma_\mu\right]\epsilon = 0, \qquad (1.85)$$

where $\nabla_\mu$ is the spinor covariant derivative on $\mathscr{M}$, $\gamma_\mu$ are the corresponding gamma-matrices, and $a^2 = C$. The property that $\mathscr{M}$ is symmetric together with Eqs. (1.56), (1.58) guarantee that $[D_\mu, D_\nu]\epsilon \equiv 0$ and (1.85) are consistent. Killing spinors are studied and used in Exercises 1.17 and 3.4.

There is an important class of field models which are invariant with respect to rescaling of fields (according to their physical dimension) and a conformal transformation of the background metric $\delta g_{\mu\nu} = \lambda g_{\mu\nu}$, where $\lambda$ is a parameter. Such models are said to be scale invariant or conformally invariant. If a theory is scale invariant it makes sense to consider also a group of symmetries which are 'isometries' up to a conformal transformation of the metric. More precisely, generators of these diffeomorphisms are solutions to the so-called conformal Killing equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \frac{2}{n}(\nabla\xi) g_{\mu\nu}. \qquad (1.86)$$

The solutions to (1.86) are called conformal Killing vectors. For example, conformal Killing vector on $\mathbb{R}^n$ is $\xi^\mu(x) = x^\mu$. We return to conformal symmetries in Chap. 8.

## 1.8 Hypersurfaces and Boundaries

The Einstein-Hilbert action (1.20) as well as action functionals discussed in Sect. 1.6 are well defined on compact closed manifolds. Physical manifolds like, for example, de Sitter, (1.30), and Schwarzschild, (1.32), solutions are not compact. To avoid integrations in the functionals over infinite regions one may "put the system in a box" or assume that the background manifold has a boundary such that the region restricted by the boundary has a finite volume. Boundaries, boundary conditions and corresponding boundary terms in the action play an important role both in classical and quantum theories, see Exercise 1.15.

If a manifold $\mathscr{M}$ has a boundary, locally in the vicinity of the boundary it looks as $\mathbb{R}^{n-1} \times \mathbb{R}_+$. The boundary is a manifold one dimension lower which we denote $\partial\mathscr{M}$. We restrict ourselves to smooth boundaries. Let us choose a coordinate system in such a way that $y^j$ are coordinates in the boundary and the remaining coordinate $z$ increases when moving from the boundary. Then the normal vector $n^\mu$ is defined by the condition $n_\mu dy^\mu = 0$, or $n_j = 0$. We shall also assume that $n$ is normalized, $n_\mu n^\mu = 1$ (for Riemannian manifolds), and that $n$ is an inward pointing. Note that $n^i = g^{i\mu} n_\mu$ may not vanish. The boundary of a Lorentzian manifold may be space-like or time-like if the normal vector is time-like or space-like, respectively. In the remaining part of this section we consider Riemannian manifolds. Generalization to boundaries in Lorentzian manifolds is straightforward.

The internal geometry of the boundary is defined by the induced surface metric $\tilde{g}_{ij} = g_{ij}$ and by its inverse $\tilde{g}^{ij}$, $\tilde{g}^{ij}\tilde{g}_{jk} = \delta^i_k$. Note that in general $\tilde{g}^{ij} \neq g^{ij}$. With the metric $\tilde{g}_{ij}$ one can construct the Christoffel symbol on the boundary, the Riemann and Ricci tensors, and the scalar curvature. To distinguish these objects from their counterparts defined in the bulk we write the boundary quantities with the tilde.

The way how the boundary $\partial\mathscr{M}$ is embedded in $\mathscr{M}$ is characterized by the extrinsic curvature (which is also called the second fundamental form of the boundary)

$$K_{\mu\nu} = -h^\lambda_\mu h^\rho_\nu n_{\lambda;\rho}, \tag{1.87}$$

where $h^\nu_\mu = \delta^\nu_\mu - n_\mu n^\nu$ is the projector on a space tangent to $\partial\mathscr{M}$ at the given point. The extrinsic curvature is symmetric, $K_{\mu\nu} = K_{\nu\mu}$, and is orthogonal to the normal, $K_{\mu\nu}n^\nu = 0$. Note, that because of the projectors in (1.87) this definition does not contain derivatives of $n_\mu$ in the normal direction (which would require an extension of $n_\mu$ to the vicinity of the boundary).

There is a very convenient coordinate system near the boundary which is called the Gaussian coordinates. In this system the coordinate $z = x^n$ measures the distance from the boundary along geodesics which are normal to the boundary. The line element then reads

$$(ds)^2 = (dx^n)^2 + g_{jk}\,dx^j\,dx^x. \tag{1.88}$$

Obviously, in this system

$$\Gamma^n_{nn} = \Gamma^n_{nj} = \Gamma^j_{nn} = 0, \qquad \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}. \tag{1.89}$$

The extrinsic curvature is given by very simple equations

$$K_{ij} = \Gamma^n_{ij} = -\frac{1}{2}\partial_n g_{ij}, \qquad K^i_j = -\Gamma^i_{nj}. \tag{1.90}$$

The remaining components, $K_{nn}$ and $K_{jn}$, vanish.

One can define covariant derivatives with respect to the metric on the boundary. We shall denote them by $\tilde{\nabla}_i$ or by the colon. For example, $\tilde{\nabla}_j v_i = v_{i:j} = \partial_j v_i - \tilde{\Gamma}^k_{ji} v_k$. The extrinsic curvature measures the difference between the covariant derivatives in the bulk and on the boundary,

$$\nabla_j v_i - \tilde{\nabla}_j v_i = -K_{ij} v_n. \tag{1.91}$$

To give an example consider a $n$-dimensional ball in $\mathbb{R}^n$ with the radius $r$. Its boundary is the $(n-1)$-dimensional sphere $S^{n-1}$. It is easy to check that the extrinsic curvature reads

$$K_{ij} = \frac{1}{r} g_{ik}. \tag{1.92}$$

In this case, $K_{ij}$ is proportional to the metric. In general, for a surface embedded in $\mathbb{R}^n$ the extrinsic curvature is a matrix constructed from the main curvature radii of the surface.

Consider two points $A$ and $B$ belonging to the boundary $\partial\mathscr{M}$ and the geodesic line $x(\tau)$ in $M$ connecting $A$ and $B$. The boundary (or a subsurface) is called totally geodesic if $x(\tau) \in \partial\mathscr{M}$ for $\tau \in [0, 1]$ and for each pair $(A, B)$ on $\partial\mathscr{M}$. The boundary is totally geodesic if and only if $K_{ij} = 0$ identically.

The extrinsic curvature is the main geometric invariant on the boundary. By means of the following Gauss-Codazzi equations one can express some other geometric quantities in terms of $K_{ij}$:

$$R^i{}_{jkl} = \tilde{R}^i{}_{jkl} - K_{jl}K^i_k + K_{jk}K^i_l, \tag{1.93}$$

$$R^n{}_{jkl} = K_{jl:k} - K_{jk:l}, \tag{1.94}$$

where $\tilde{R}^i{}_{jkl}$ is the Riemann tensor constructed from the boundary metric.

Let us now explain how one can construct invariants associated with the boundary. The general recipe is to take a scalar on the boundary and to integrate it over $\partial\mathscr{M}$ with the weight $\sqrt{\det g_{ij}}$. The scalars can be constructed by contracting all indices $i, j, k$ etc. of arbitrary tensors on the boundary and taking the trace over all bundle (gauge and spin) indices. This recipe looks precisely as the one for constructing the bulk invariants, except that on the boundary we have much more tensors. In addition to all tensors existing in the bulk, one can also construct tensors from the extrinsic curvature and its derivatives. Note, that since $K_{ij}$ is defined on the boundary only, one can differentiate it only tangentially. Normal derivative of the extrinsic curvature makes no sense. Due to (1.93) the boundary Riemann tensor is not an independent quantity. Note, that normal indices need not be contracted. For example, $\int_{\partial\mathscr{M}} d^{n-1}x \sqrt{\det g_{ij}} R_{;n}$ and $\int_{\partial\mathscr{M}} d^{n-1}x \sqrt{\det g_{ij}} \operatorname{tr} \Omega_{jn} K_{ik:l} \tilde{g}^{ji} \tilde{g}^{kl}$ are allowed invariants.

## 1.9 Defects of Geometry

In various physical examples one may encounter background spaces which are smooth everywhere except some hypersurfaces $\Sigma$ located *inside* the space. Outside $\Sigma$ the space can be considered as a manifold. Because the tangent space is not defined on $\Sigma$ the curvature cannot be defined in this region as well. There are however the situations when the curvature characteristics cannot be defined locally (either because of discontinuities or as a result of topological obstructions) while integrals of curvature invariants still have a meaning.

In this case we call $\Sigma$ a defect of the geometry. Below we give two most important examples of defects: when $\Sigma$ is a codimension one or codimension two hypersurface.

**Codimension One Defects and Branes** Singularities on hypersurfaces of dimension $(n-1)$ located *inside* the manifold are called branes (from membranes). From the mathematical point of view, branes occur when one glues together two smooth manifolds, $\mathscr{M}_+$ and $\mathscr{M}_-$, along their common boundary $\Sigma = \partial\mathscr{M}_+ = \partial\mathscr{M}_-$. Obviously the geometric quantities induced on the brane from $\mathscr{M}_+$ and $\mathscr{M}_-$ which define internal geometry of $\Sigma$ (such as the metric, connection along the brane, etc.) must agree, but other quantities may not. One of the quantities which may jump is the extrinsic curvature. The smooth situation corresponds to $K^+_{ij} = -K^-_{ij}$ (the minus sign appears due to opposite orientations of the inward pointing normals in $\mathscr{M}_+$

and $\mathscr{M}_-$). The jump of the extrinsic curvature is therefore $K^+_{ij} + K^-_{ij}$, and it is determined by the matter distribution on $\Sigma$. The corresponding equation is called the Israel junction condition [162] (see Exercise 1.16). An example of such situation is the brane-world metric

$$(ds)^2 = (dx^5)^2 + e^{\alpha|x^5|}(ds_4)^2, \tag{1.95}$$

where $\alpha$ is a real constant. A four-dimensional brane with the line element $(ds_4)^2$, is located at $x^5 = 0$.

A more down to earth situation when singularities appear on hypersurfaces is of a non-geometric origin. It correspond to singular potentials, like delta functions, concentrated on hypersurfaces. Such potentials describe semi-transparent boundaries and occupy a position between smooth potentials and hard (non-penetrable) boundaries.

**Conical Singularities** A well-known type of codimension two defects on Riemannian manifolds are conical singularities. It is instructive to start discussion of such manifolds with a two-dimensional metric

$$ds^2 = d\rho^2 + \rho^2 d\tau^2, \tag{1.96}$$

where $0 < \rho < \infty$ and $\tau$ be a cyclic coordinate, $0 < \tau \leq \beta$. If $\beta = 2\pi$, the line element represents the metric on a two-dimensional plane written in the polar coordinates. If $\beta \neq 2\pi$, one deals with a two-dimensional cone. We denote the cone $C_\beta$. The tip of the cone, $\rho = 0$, is a singular point because the length of a circle with the center at the tip and radius $\rho$ does not equal $2\pi\rho$.

The conical space has a non-trivial curvature which behaves as a distribution at the tip. To see this it is convenient to consider another metric

$$ds^2 = d\theta^2 + \sin^2\theta\, d\tau^2, \tag{1.97}$$

where $0 \leq \theta \leq \pi$. If $\tau$ ranges from 0 to $2\pi$ the metric (1.97) defines the line element on the sphere $S^2$. If $0 < \tau \leq \beta$ and $\beta \neq 2\pi$ the space (1.97) has conical singularities at the "north" and "south" poles ($\theta = 0, \pi$). We denote this space $S^2_\beta$. One can define $S^2_\beta$ as a limit of a sequence of smooth closed 2-spaces with the topology of $S^2$. This assumption implies that the Euler characteristics (1.29) of $S^2_\beta$ equals 2. By using (1.29) for $p = 1$ one can write for $S^2_\beta$

$$\int_{S^2_\beta} \sqrt{g}\, d^2x\, R = 4\pi\chi_1[S^2_\beta] = 8\pi. \tag{1.98}$$

The integral of the curvature consists of the two pieces:

$$\int_{S^2_\beta} \sqrt{g}\, d^2x\, R = 4\beta + I_{\text{sing}}. \tag{1.99}$$

The first term in the r.h.s. of (1.99) comes from points outside the poles where the scalar curvature is standard, $R = 2$. The second term, $I_{\text{sing}}$, is due to singular behavior of the curvature at the poles. By comparing (1.98) and (1.99) one concludes that $I_{\text{sing}} = 4(2\pi - \beta)$. Thus, each conical singularity with coordinates $x_s$ yields a delta function contribution to the curvature

$$R_{\text{sing}}(x) = 2(2\pi - \beta)\frac{1}{\sqrt{g}}\sum_{x_s}\delta^{(2)}(x - x_s). \tag{1.100}$$

In physical applications one can meet a general type of manifolds with conical singularities. These are manifolds which possess internal co-dimension two hypersurfaces $\Sigma$ such that in the vicinity of $\Sigma$ a manifold has the structure $C_\beta \times \Sigma$.

In many problems there may exist a global Killing vector field $\partial_\tau$ on a manifold such that conical singularities are fixed points of $\partial_\tau$. Manifolds with this property make a family of spaces denoted by $\mathscr{M}_\beta$ and parametrized by $\beta$. All representatives of a family have an identical local geometry outside $\Sigma$. There is a member of the family, $\mathscr{M} = \mathscr{M}_{2\pi}$, which does not have conical singularities. One can use $\mathscr{M}$ to define a pair of vector fields $n_i$ which are orthogonal to $\Sigma$ and normalized as $(n_i \cdot n_j) = \delta_{ij}$, and determine extrinsic geometrical invariants of $\Sigma$. Due to the $O(2)$ isometry extrinsic curvatures of $\Sigma$ vanish. Invariants of the other type are

$$R_{ii} \equiv \sum_i R_{\mu\nu} n_i^\mu n_i^\nu, \qquad R_{ijij} \equiv \sum_{ij} R_{\mu\lambda\nu\rho} n_i^\mu n_j^\lambda n_i^\nu n_j^\rho, \tag{1.101}$$

where $R_{\mu\nu}$ and $R_{\mu\lambda\nu\rho}$ are the components of the Ricci and Riemann tensors of $\mathscr{M}$ at $\Sigma$. The quantities (1.101) are $O(2)$ invariant, i.e. do not depend on orientation of $n_i$. There are other similar invariants on $\Sigma$ of the same dimensionality. For instance, the scalar curvature $R$ of $\mathscr{M}$ and a scalar curvature $R_\Sigma$ of $\Sigma$ itself. The Gauss-Codazzi equation, however, tells that

$$R_\Sigma = R - 2R_{ii} + R_{ijij}. \tag{1.102}$$

Thus, the only independent invariants of the lowest dimensionality are $R_{ii}$, $R_{ijij}$, and $R$. We will use these quantities in Sect. 4.7 when studying spectral geometry on base manifolds with conical singularities.

## 1.10 Literature Remarks

There is a number of excellent introductions in the physical and mathematical foundations of the general relativity theory. It is enough to mention classical monographs by Weinberg [252], Misner, Thorne and Wheeler [191], and Synge [235]. A more elaborate discussion of geometrical formulations used in Sects. 1.2, 1.5 can be found in Eguchi, Gilkey and Hanson [99]. Other recommended books are [164, 175, 196].

The most complete source of information about classical and quantum properties of black holes is the monograph by Frolov and Novikov [114].

For an introduction to group theory motivated by application to quantum field theory we refer to the book by Barut and Raczka [25].

A useful introduction to the Killing equations and symmetric manifolds is presented in [252]. Here one can find a proof of a number of statements made in Sect. 1.7.

The role of boundary terms both in classical and quantum gravity has been emphasized by York [260] and by Gibbons and Hawking [130].

The 'defects of geometry' discussed in Sect. 1.9 are the geometrical notions motivated by physical models. The branes are the objects which are used in the non-Einstein theories of gravity. The idea is that the four-dimensional world may be a domain-wall (a brane) in higher-dimensional space-time. The braneworlds models which appeared in [212, 213, 220] allow short-distance modification of the Einstein theory and predict a new scale of quantum gravity effects.

Codimension 2 defects received much attention first in connection with cosmic strings [250], hypothetical objects whose creation during the phase transitions in the early universe is predicted by grand unification models. Another example are solutions to the Einstein equations which describe configurations of several black holes. These configurations can be made static if black holes are connected by struts which result in conical singularities. It is also worth mentioning that conical defects appear in the so-called off-shell formulation of black hole thermodynamics, see e.g. [114]. Two-dimensional manifolds with a finite number of conical singularities are called *thornyfolds* [115]. More general singular manifolds are called *conifolds* [222]. Conifolds have singularities with the structure of higher-dimensional cones and appear in applications to string theory.

Recommended Exercises are 1.7–1.11, 1.13, and 1.17.

## 1.11 Exercises

**Exercise 1.1** Find transformation properties of the Christoffel connection $\Gamma^\rho_{\mu\nu}$, see (1.4), under a change of the coordinates.

**Exercise 1.2** Prove that on a Riemannian manifold the object $\varepsilon^{\mu_1\cdots\mu_n} = g^{-1/2}\tilde{\varepsilon}^{\mu_1\cdots\mu_n}$ is a tensor. Here $\tilde{\varepsilon}^{\mu_1\cdots\mu_n}$ is the totally antisymmetric Levi-Civita symbol such that $\tilde{\varepsilon}^{12\ldots n} = 1$, $g = \det g_{\mu\nu}$.

**Exercise 1.3** Calculate the Riemann tensor, the Ricci tensor, and the scalar curvature on a two-sphere of the radius $r$. Make sure that

$$R_{\mu\nu\rho\sigma} = r^{-2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{1.103}$$

**Exercise 1.4** Prove the following variational formulae:

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu}, \tag{1.104}$$

$$\delta R^\sigma_{\ \mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^\sigma_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\sigma_{\mu\lambda}), \tag{1.105}$$

$$\delta R_{\mu\nu} = \frac{1}{2}\big[\nabla^\lambda\nabla_\nu\delta g_{\lambda\mu} + \nabla^\lambda\nabla_\mu\delta g_{\lambda\nu} - \nabla^\lambda\nabla_\lambda\delta g_{\mu\nu} - \nabla_\nu\nabla_\mu\big(g^{\lambda\sigma}\delta g_{\lambda\sigma}\big)\big], \tag{1.106}$$

$$\delta R = -\delta g_{\mu\nu}R^{\mu\nu} + \big[\nabla^\lambda\nabla^\mu\delta g_{\lambda\mu} - \nabla^\lambda\nabla_\lambda\big(g^{\mu\nu}\delta g_{\mu\nu}\big)\big]. \tag{1.107}$$

**Exercise 1.5** Check that the Schwarzschild metric (1.32) takes the form (1.35) in the Eddington-Finkelstein coordinates $r$, $v = t + r^*$ where $r^*$ is defined by (1.34).

**Exercise 1.6** Find the stress-energy tensor of the vector field model (1.75).

**Exercise 1.7** Use the definitions of the Riemann tensor, (1.10), and the connection, (1.51), to prove the relation

$$\partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\nu{}^{ab} + \omega_\mu{}^a{}_c \omega_\nu{}^{cb} - \omega_\mu{}^a{}_c \omega_\mu{}^{cb} = R^{ab}{}_{\mu\nu},$$

where $R^{ab}{}_{\mu\nu} = e^a_\lambda e^b_\rho R^{\lambda\rho}{}_{\mu\nu}$.

**Exercise 1.8** Check that $\gamma$-matrices obey the following rule of Hermitian conjugation

$$\beta(\gamma_\mu)^\dagger \beta = -\gamma_\mu, \tag{1.108}$$

where $\beta = i\gamma^{a=0}$.

**Exercise 1.9** Find an expression for the generators of $Spin(1, n-1)$ in terms of the gamma-matrices

**Exercise 1.10** Prove that in the Lorentzian theory there is the following relation for the matrices of the $Spin(1, n-1)$ group:

$$\beta S^\dagger \beta = S^{-1}.$$

**Exercise 1.11** Consider the Dirac spinor on a four-dimensional Lorentzian manifold. Find a matrix $C$ which is used to define the charge conjugated spinor $\psi^c$, see Eqs. (1.66), (1.67). Prove that $\psi$ and $\psi^c$ belong to equivalent representations of the spin group.

**Exercise 1.12** Show that the Majorana fermions defined at the end of Sect. 1.5 cannot interact with vector fields. Show that if the Majorana spinor were taken a commuting field, it had to be massless.

**Exercise 1.13** Consider a two-dimensional spherical cap which is obtained from the two-sphere by restricting the angle $\theta$ in (1.33) to the interval $[0, \theta_0]$. Calculate the extrinsic curvature. Make sure that for $\theta_0 = \pi/2$ the boundary is totally geodesic.

**Exercise 1.14** Calculate the extrinsic curvatures $K^+_{ij}$ and $K^-_{ij}$ for the brane-world metric (1.95).

**Exercise 1.15** Consider the Einstein-Hilbert action (1.20) on a Lorentzian manifold $\mathscr{M}$ with a boundary $\partial\mathscr{M}$. As follows from (1.107) variations of the action result in boundary terms which contain variations of the metric tensor $\delta h_{\mu\nu}$ on $\partial\mathscr{M}$ and normal derivatives of variations of the metric at the boundary.

Find the variation of the following functional in the presence of boundary terms:

$$\tilde{I}_{EH}[g] = \frac{1}{16\pi G_N}\int_{\mathscr{M}} d^n x\,\sqrt{-g}(R-2\Lambda) + \frac{1}{8\pi G_N}\int_{\partial\mathscr{M}} d^{n-1}x\,\sqrt{h}K. \tag{1.109}$$

Check that boundary terms with normal derivatives do not appear and, hence, this functional is extremal on solutions to the Einstein equations provided that metric on $\partial\mathscr{M}$ is fixed. The form of the action (1.109) was suggested by Gibbons and Hawking [130].

**Exercise 1.16** Consider a brane (see Sect. 1.9) which is a codimension one defect of the geometry. By using results of the previous problem derive the Israel junction condition

$$(K^{\lambda\mu} - h^{\lambda\mu}K)_+ + (K^{\lambda\mu} - h^{\lambda\mu}K)_- = 8\pi G_N \mathscr{T}^{\mu\nu}. \tag{1.110}$$

The l.h.s. of this relation describes the jump of extrinsic curvature on the brane, $\mathscr{T}^{\mu\nu}$ is the stress-energy tensor of the matter localized on the brane

$$\mathscr{T}^{\mu\nu} = \frac{2}{\sqrt{h}}\frac{\delta I_{\text{brane}}[\varphi,h]}{\delta h_{\mu\nu}}, \tag{1.111}$$

$I_{\text{brane}}[\varphi,h]$ is the corresponding action of the matter localized on the brane, and $h_{\mu\nu}$ the metric induced on the brane.

**Exercise 1.17** Show that the Killing equation

$$\left(\nabla_\mu + \frac{i}{2}\gamma_\mu\right)\epsilon = 0, \tag{1.112}$$

on the unit two-sphere (1.97) admits two independent solutions. Use the Killing spinors to construct the Killing vectors on $S^2$.

# Chapter 2
# Quantum Fields

## 2.1 Relativistic Inner Product

In this Chapter we start a systematic discussion of a quantum theory in external classical background fields. First, we introduce a specific inner product and define quantization conditions based on this product.

The method of quantization which we present below is not a substitute for more profound methods, like the full Hamiltonian analysis or the so-called BRST approach, but it allows one to arrive faster to the results in the lowest order of the perturbation theory on non-trivial backgrounds. In this section, we shall be rather sloppy with a mathematical side of the statements, ignore all the functional analysis issues, for example, and simply use a finite-dimensional intuition in infinite-dimensional spaces of fields.

Let $\varphi$ be a non-interacting field on a Lorentzian space-time $\mathscr{M}$. We call $\varphi$ a dynamical variable to distinguish it from the background. It is assumed that $\varphi$ belongs to a section in a fiber bundle over $\mathscr{M}$. We further assume that $\mathscr{M}$ is a globally hyperbolic space-time. The dimensionality $n$ of $\mathscr{M}$ is not fixed ($n \geq 2$).

Let us choose on $\mathscr{M}$ some coordinate system $x^\mu \equiv (t, x^k)$, where $k = 1, \dots, d$ and $d = n - 1$. Here, we assume existence of a foliation $\mathscr{M}$ by spatial sections, so that at least locally the space-time manifold looks as a direct product of a one-dimensional "time" and a $d$-dimensional "space". As we saw (see Sect. 1.6), small fluctuations $\varphi$ obey linearized equations of motion of the form

$$P(\partial_t, \partial_k)\varphi(t, x^k) = 0, \tag{2.1}$$

where for integer spin fields $P(\partial_t, \partial_k)$ is a second order hyperbolic type partial differential operator. For spin $1/2$ fields $P(\partial_t, \partial_k)$ is a first order operator, see (1.74).

Let $f_1$ and $f_2$ be a pair of solutions to (2.1). We are going to introduce a so-called *relativistic* inner product, $\langle f_1, f_2 \rangle$, between these solutions. The product is constructed through a conserved current corresponding to some global symmetry. Since the inner product and the current must depend on *two* fields instead of one, we have to double the number of fields in the quadratic form of the action. This is

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done in the following way. Consider first the case of a complex field and write the quadratic action which generates the linearized equations (2.1) as

$$I_2[\varphi] = \int d^n x \sqrt{-g}\, \varphi^* P \varphi, \tag{2.2}$$

where all vector or gauge indexes (if any) are suppressed. An example of a functional which can be brought to this form (after integrating by parts) is the scalar action (1.68). Although (2.2) is a rather typical form of the quadratic action it is not universal. In some cases, for instance, for an action of small fluctuations in the $(\varphi\varphi^*)^2$ model, there may also appear $(\varphi^*)^2$ and $\varphi^2$ terms. We shall comment how to deal with such models in the end of this section.

Next, we go from (2.2) to a sesquilinear form

$$I[f_1, f_2] = \int d^n x \sqrt{-g}\; f_1^* P f_2 \equiv \int d^n x \sqrt{-g} L(f_1^*, f_2) \tag{2.3}$$

such that $I_2[\varphi] = I[\varphi, \varphi]$. We assume that the operator $P$ is at least formally self-adjoint (see Sect. 3.1), therefore the form is Hermitian. The sesquilinear form $I[f_1, f_2]$ is linear in the second argument and antilinear in the first. The quantities $I[f_1, f_2]$, $L(f_1^*, f_2)$ can be considered as a field theory action and a Lagrange density, respectively. Indeed, one gets for $f_k$ the same equations of motions (2.1) by requiring that variations $I[f_1, f_2]$ over $f_k$ have to vanish. If $P$ is a second order operator we shall always assume that second derivatives in $I[f_1, f_2]$ are eliminated by integrating by parts and $L(f_1^*, f_2)$ contains at most first derivatives of $f_1$ and $f_2$.

The functional $I[f_1, f_2]$ has an obvious global symmetry $\{f_1, f_2\} \to \{e^{i\alpha} f_1, e^{i\alpha} f_2\}$ which implies the existence of a conserved current. To derive this current, consider an infinitesimal version of the transformations

$$\delta_\alpha f_1^* = -i\alpha f_1^*, \qquad \delta_\alpha f_2 = i\alpha f_2 \tag{2.4}$$

and assume for a moment that the transformation parameter $\alpha$ depends on the coordinates, $\alpha \to \alpha(x)$. Then, transformations (2.4) are no longer symmetries of the action. Nevertheless, the variation of (2.3) vanishes on constant $\alpha$ and, hence, is proportional to the derivative of $\alpha$,

$$\begin{aligned}\delta_\alpha I[f_1, f_2] &= -\int d^n x \sqrt{-g} (\partial_\mu \alpha) \cdot j^\mu[f_1, f_2] \\ &= \int d^n x \sqrt{-g} \alpha \cdot \nabla_\mu j^\mu[f_1, f_2]\end{aligned} \tag{2.5}$$

for some current $j^\mu$. Next, suppose that $f_1$ and $f_2$ are solutions to the classical field equations. Then, any infinitesimal variation of the action vanishes, including the one given in (2.4) with arbitrary local parameter $\alpha$. In other words, on shell $\delta_\alpha I = 0$ for any alpha, and the current $j^\mu$ is conserved

$$\nabla_\mu j^\mu (f_1, f_2) = 0. \tag{2.6}$$

The arguments presented above also provide us with a method to compute the conserved current. For bosonic theories with actions depending on the first derivatives at most, one can easily show, that

$$j^\mu(f_1, f_2)\alpha = \frac{\partial L(f_1^*, f_2)}{\partial f_{2,\mu}}\delta_\alpha f_2 + \frac{\partial L(f_1^*, f_2)}{\partial f_{1,\mu}^*}\delta_\alpha f_1^* \tag{2.7}$$

or

$$j^\mu(f_1, f_2) = i\left(\frac{\partial L[f_1^*, f_2]}{\partial f_{2,\mu}} f_2 - \frac{\partial L[f_1^*, f_2]}{\partial f_{1,\mu}^*} f_1^*\right). \tag{2.8}$$

These statements are a particular case of what is known as the Noether theorem for global symmetries. We shall also deal with the Noether theorem for local symmetries in Chap. 8. The current $j^\mu(f_1, f_2)$ is called the Noether current.

Let us take a space-like hypersurface $\Sigma$ in $\mathscr{M}$ and construct the following inner product between the classical solutions

$$\langle f_1, f_2\rangle = \int_\Sigma d\Sigma^\mu\, j_\mu(f_1, f_2), \tag{2.9}$$

which is called the *relativistic inner product*. The product is linear in the second argument $f_2$ and anti-linear in $f_1$, and is Hermitian, $\langle f_1, f_2\rangle = \langle f_2, f_1\rangle^*$. Here $d\Sigma^\mu = n^\mu \det h\, d^d x$, $n^\mu$ is a unit future directed vector orthogonal to $\Sigma$, $\det h\, d^d x$ is the invariant measure on $\Sigma$. The continuity property (2.6) ensures that the product $\langle f_1, f_2\rangle$ does not depend on local deformations of $\Sigma$.

We call $\langle f_1, f_2\rangle$ a relativistic product to distinguish it from another inner product between sections of the fiber bundles defined in Sect. 1.5. Let us give now a couple of examples.

**Charged Scalar Field** For the model described by the action (1.68) we assume that the gauge field is a background field, and $\varphi$ is dynamical, and obtain

$$j_\mu(f_1, f_2) = i(f_1^* D_\mu f_2 - (D_\mu f_1)^* f_2). \tag{2.10}$$

The continuity equation (2.6) can be checked directly. It follows from the identity $\nabla^\mu(f_1^* D_\mu f_2) = (D^\mu f_1)^* D_\mu f_1 + f_1^*(D^\mu D_\mu f_2)$ and Eq. (1.69). The relativistic product constructed from this current is called the Klein-Gordon product.

**Spinor Fields** Consider the model (1.74). By repeating the computations presented above and taking care of the order of fields, one arrives at

$$j_\mu(\psi_1, \psi_2) = -i\bar{\psi}_1\gamma_\mu\psi_2. \tag{2.11}$$

For the case when the field $\varphi$ is *real*, instead of (2.2) one has the functional

$$I_2[\varphi] = \frac{1}{2}\int d^n x\,\sqrt{-g}\,\varphi P\varphi. \tag{2.12}$$

The corresponding complex Hermitian sesquilinear form is

$$I_R[f_1, f_2] = \frac{1}{2}\int d^n x\,\sqrt{-g}\,f_1^* P f_2. \tag{2.13}$$

(Notice the difference between real and complex field actions in the coefficient $1/2$.) The Noether current which is used to construct the relativistic product is given by (2.8). Examples involving real fields are as follows.

**Vector Fields** For the model described by equation of motion (1.76) after the replacement $f_k \to A_k^\mu$ one finds

$$j_\mu(A_1, A_2) = i(A_1^\nu)^* F_{\mu\nu}(A_2) - i A_2^\nu (F_{\mu\nu}(A_1))^*. \tag{2.14}$$

The same product holds for the gauge potential in the Maxwell theory.

**Linearized Yang-Mills Theory** For the theory described by the linearized equations (1.79)

$$j_\mu(A_1, A_2) = -2i\,\mathrm{Tr}((A_1^\nu)^+ G_{\mu\nu}(A_2) - (G_{\mu\nu}(A_1))^+ A_2^\nu), \tag{2.15}$$

where the tensor $G_{\mu\nu}$ is defined in (1.80).

The following useful permutation property of the product can be now inferred from (2.10), (2.11), (2.14), (2.15):

$$\langle f_1, f_2\rangle = \pm\langle f_2^*, f_1^*\rangle. \tag{2.16}$$

Here the plus sign in the r.h.s. corresponds to the (non-Grassmann) Dirac fields and the minus sign stands for scalar and vector fields. The sign, thus, depends on whether the spin is integer or half-odd-integer. For spinor fields the star operation in (2.16) can be replaced with the charge conjugation defined in (1.66), see Exercise 2.7.

As we have already mentioned above, not any quadratic action depending on complex fields can be represented through a sesquilinear form. To overcome this difficulty one has to introduce independent real fields as real and imaginary parts of the original complex field, diagonalize the action and then complexify it as we have just described. As a result, one obtains a conserved current, but the number of complex degrees of freedom is twice the original one.

## 2.2 Quantization and Single-Particle Excitations

To set the stage for the quantization we start with anti-linear functionals acting on classical solutions. Each such functional can be constructed as $\varphi[f] = \langle f, \varphi\rangle$, where $\varphi$ is some *fixed solution* to the classical equations. It is allowed to multiply these functionals calculated for several solutions with different arguments. In this way, one obtains multilinear functionals. One can as well define complex conjugated linear functionals $\varphi^+[f] = \langle\varphi, f\rangle = (\varphi[f])^*$ and introduce 'real' functionals which obey the restriction $\langle\varphi, f\rangle = \langle\varphi^*, f\rangle$. With the help of (2.16) the reality condition can be written as

$$\varphi^+[f] = \pm\varphi[f^*], \tag{2.17}$$

where the plus or minus signs correspond to spin $1/2$ or spins 0 and 1, respectively. The star operation for spin $1/2$ fields denotes the charge conjugation (1.66). In the case of spin $1/2$ fields, the classical solutions will be considered as commuting (classical) spinors, while the functionals $\varphi$ will anticommute before the quantization.

Quantization means that to each classical solution $f$ one puts into a correspondence an operator $\varphi[f]$ and its Hermitian conjugate $\varphi^+[f]$. These operators act on vector spaces, the so-called Fock spaces discussed below. The operators $\varphi[f]$, $\varphi^+[f]$ are operator-valued distributions, an analog of classical functionals defined above. Thus, they are denoted by the same letter. Like the classical functionals, $\varphi[f]$, $\varphi^+[f]$ are, respectively, anti-linear or linear in their arguments. It is also required that operators preserve symmetry properties of the classical functionals.

The operators are required to obey the following *quantization conditions*:

$$[\varphi[f_1], \varphi^+[f_2]]_\pm \equiv \varphi[f_1]\varphi^+[f_2] \pm \varphi^+[f_2]\varphi[f_1] = \hbar\langle f_1, f_2\rangle, \tag{2.18}$$

where the parameter $\hbar$ is the Planck constant. Starting with Sect. 2.6 we shall put $\hbar = 1$. For integer spin fields one uses the commutator $[,]_-$ and says that the fields obey the Bose statistics, for half-odd-integer spins one uses anti commutator $[,]_+$ which implies the Fermi statistics. The quantization condition (2.18) is fully covariant, it does not depend on the choice of coordinates and the Cauchy surface used. The features of a particular model which is quantized are encoded in the relativistic product and properties of the classical solutions $f_k$.

Classically, the bosonic field functionals commute, while the fermionic ones anticommute. Quantization means that we *deform* these simple (anti-)commutation relations by adding a non-zero right hand side to relation (2.18). The Plank constant $\hbar$ plays the role of a deformation parameter.

One can define Hermitian operators by condition (2.17). These operators corresponds to real fields. Quantization in this case is determined by the same rule (2.18).

As a next step one has to consider the two problems: to find an operator analog of a local field and to describe elementary field excitations. The second task is motivated by the fact that a free field theory can be interpreted as a system of infinitely many oscillators. We have to find a way how to decouple different oscillations and introduce the corresponding creation and annihilation operators by following the quantum mechanical example.

To solve the two problems we need a basis which brings the relativistic inner product to a canonical form. In this section, it is convenient to consider models where *the relativistic product is non-degenerate*, i.e. if $\langle f_1, f_2\rangle = 0$ for all $f_2$, then $f_1 = 0$. An important example of theories with the degenerate product are gauge theories. They will be considered in Sect. 2.3.

When the relativistic product is non-degenerate it can be diagonalized by introducing a basis $\{f_A\}$, so that $\langle f_A, f_B\rangle = \lambda_A\delta_{AB}$. Because of the hermiticity of the inner product, the eigenvalues $\lambda_A$ are real. By a suitable rescaling one can make these eigenvalues equal to $\pm 1$. This yields a set of modes $\{f_i^{(+)}, f_j^{(-)}\}$ which satisfies the following conditions:

$$\langle f_i^{(+)}, f_j^{(-)}\rangle = 0, \tag{2.19}$$

$$\langle f_i^{(\pm)}, f_j^{(\pm)}\rangle = \pm\delta_{ij}, \quad \text{for Bose statistics,} \tag{2.20}$$

$$\langle f_i^{(\pm)}, f_j^{(\pm)}\rangle = \delta_{ij}, \quad \text{for Fermi statistics.} \tag{2.21}$$

Here $\delta_{ij}$ is the Kronecker symbol if $i, j$ are discrete indices, and it is a delta-function if $i, j$ take continuous values.

We call $f_i^{(\pm)}$ the *single-particle* modes. In the case of Bose fields the relativistic product is not positive-definite, and the modes $f^{(+)}$, $f^{(-)}$ have positive or negative norm, respectively. In the case of Fermi fields the product is positive, see Exercise 2.7. The division on "+" and "−" modes in this case is related to other properties, for example, to the sign of the frequency carried by the mode in stationary or asymptotically stationary space-times, see details in Sect. 2.5. In certain cases the "−" spin $1/2$ modes can be also defined as charge conjugated "+" modes, see below.

Any solution to field equations (2.1) can be uniquely represented as a linear combination of $f_i^{(+)}$ and $f_j^{(-)}$,

$$f(x) = \sum_i c_i f_i^{(+)}(x) + \sum_j d_j f_j^{(-)}(x), \tag{2.22}$$

where $c_i$ and $d_j$ are some complex numbers which can be determined with the help of the normalization conditions (2.19)–(2.21), $c_i = \langle f_i^{(+)}, f\rangle$, $d_j = \mp\langle f_j^{(-)}, f\rangle$. If $i$ and $j$ take continuous values the sums in (2.22) correspond to integrals.

Local field operators can be defined by analogy with (2.22). First one introduces the operators

$$a_i = \varphi[f_i^{(+)}], \qquad b_i^+ = \mp\varphi[f_i^{(-)}], \tag{2.23}$$

called the *annihilation* and *creation* operators, respectively. In the definition of $b_i^+$ the minus sign corresponds to the Bose statistics, the plus sign is for the Fermi statistics. By using (2.22), (2.23) and the assumption that $f_i^{(\pm)}$ is a complete set of modes the operator functionals $\varphi[f]$ can be represented as

$$\varphi[f] = \sum_i c_i a_i + \sum_j d_j b_j^+. \tag{2.24}$$

The *local* operator of a quantized field is then defined as

$$\varphi(x) = \sum_i a_i f_i^{(+)}(x) + \sum_j b_j^+ f_j^{(-)}(x). \tag{2.25}$$

This formula together with (2.24) allows one to write the operator functionals in the form, $\varphi[f] = \langle f, \varphi\rangle$, where the local operator (2.25) appears as an argument in the product. The important feature of the quantized field operator $\varphi(x)$ is that it is a formal solution to the field equations (2.1). Due to this property, Eq. (2.25) is a key formula for computing quantum averages.

The creation and annihilation operators (2.23) solve the problem of an oscillator representation of a free field theory. Indeed, by using (2.18) and normalization conditions (2.19)–(2.21) one arrives at the following commutation relations:

$$[a_i, a_j^+]_\pm = \hbar\delta_{ij}, \qquad [b_i, b_j^+]_\pm = \hbar\delta_{ij} \tag{2.26}$$

(commutators between $a_i$ and $b_j$ vanish). Apart from a different meaning of the indices $i$ and $j$, these commutators are identical to those appearing in quantum mechanics of harmonic oscillator.

Formal polynomials of creation and annihilation operators modulo relation (2.26) form an associative algebra. It can be represented by linear operators acting on the Fock space, which can be introduced in the following way. First, one takes a special vector $|0\rangle$ such that

$$a_i|0\rangle = b_i|0\rangle = 0, \tag{2.27}$$

for all annihilation operators. It is called the vacuum vector or the ground state. Other vectors which constitute a basis in the Fock space are obtained by acting on $|0\rangle$ by all possible monomials of the creation operators,

$$|i_1, \dots i_k, j_1 \dots j_p\rangle = C_{i_1,\dots i_k, j_1 \dots j_p}(a^+_{i_1})^{n_1} \dots (a^+_{i_k})^{n_k}(b^+_{j_1})^{m_1} \dots (b^+_{j_p})^{m_p}|0\rangle, \tag{2.28}$$

where $C_{i_1,\dots i_k, j_1 \dots j_p}$ are normalization coefficients. These states describe fields excitations with a fixed number of quanta.

There can be infinitely many different ways to specify field excitations and to choose a set of single-particle modes $f_i^{(\pm)}$. This also implies that the ground state is not universal. The ground state with respect to one set of quanta may look as a state containing quanta defined in a different way. Indices $i$ and $j$ describe quantum numbers such as, for example, spin of the quanta and the momentum in a certain frame of reference. Thus, the choice of modes is determined by physical characteristics of the systems which are measured. The different sets of creation and annihilation operators are related to each other by unitary transformations called the Bogoliubov transformations, see Exercise 2.2.

At the end of this section a comment on quantum theory of real fields is in order. The equations of motion for real fields are invariant with respect to the complex conjugation or the charge conjugation (as in case of spin 1/2 fields). By taking into account (2.16) and conditions (2.19)–(2.21) one can conclude that $(f_j^{(-)})^* = f_j^{(+)}$. Since the corresponding operators are Hermitian, Eqs. (2.17) and (2.23) show that operators $a_i$ and $b_i$ coincide and just one set of these operators, say $a_i$ and $a_i^+$, is used in this case.

## 2.3 Comments on Gauge Fields

Consider now theories with a degenerate relativistic inner product. The degeneracy means that there are classical solutions $\xi$ for which the product with any other solution $f$ vanishes identically, $\langle f, \xi\rangle \equiv 0$. Such a situation happens in theories where gauge fields are dynamical variables and we call $\xi$ gauge modes. The examples are the Maxwell and Yang-Mills models (the model (1.75) for $M = 0$ and the model (1.77), respectively). In both models the classical action and equations of motion are invariant with respect to the gauge transformations $\delta_\xi f = \xi$. For Yang-Mills fields and other fields with non-linear dynamics this property applies to small perturbations which are described by linear equations.

The gauge modes are unphysical degrees of freedom because they do not contribute to physical quantities. In contrast, one can define *physical modes* as solutions $f$ with a non-vanishing norm $\langle f, f\rangle \neq 0$. Modes related by a gauge transformation, $f$ and $f_\xi = f + \xi$, are physically equivalent. One says that they belong the same orbit of the gauge group.

Consider classical functionals introduced in Sect. 2.2. In the case of gauge theories, let us require that $\varphi[f]$ acts on a set of all physical modes and does not vanish identically on this set. By their definition, the functionals $\varphi[f]$ are gauge invariant, $\varphi[f] = \varphi[f_\xi]$, thus, one can also say that they are defined on the orbits of the gauge group.

When going to quantum theory one replaces classical functionals with operator functionals $\varphi[f]$ also acting on the orbits. Introduction of the gauge invariant operators is justified because the r.h.s. of the commutation relations (2.18) is gauge invariant. Such quantization approach can be called "quantization in physical modes".

Instead of working with an orbit it is more convenient to choose one of its representatives, a particular mode by requiring that the mode obeys certain conditions. This is called a gauge fixing procedure. The fact that gauge conditions eliminate the gauge freedom implies that their solutions intersect each orbit of the gauge group in exactly one point. Generically, such conditions cannot be chosen globally on the whole space of the fields, but, since we are working with small fluctuations only, it is not a problem.

Let us illustrate the method by using a pure Maxwell theory. The gauge modes here have the simple form, $\xi_\mu = \partial_\mu \lambda$, where the gauge parameter $\lambda$ is a sufficiently smooth function. For any potential $A_\mu$ there is a gauge parameter $\lambda$ such that after the corresponding transformation the potential satisfies the so-called *Lorentz condition* $\nabla^\mu A_\mu = 0$. This condition does not eliminate the gauge freedom completely because it is invariant under the transformations where the gauge parameter is a solution to equation $\nabla^2 \lambda = 0$. This extra freedom is fixed by requiring that some components of the potential are vanishing, for example, that $A_0 = 0$ (on classical solutions). This means that the number of physical degrees of freedom of a photon in $n$ dimensions is $n - 2$. For a theory in Minkowski space-time the above conditions can be written as $A_0 = \partial^i A_i = 0$. This confirms the fact that the physical degrees of a photon are two polarizations orthogonal to the spatial momentum.

Once the gauge is fixed and physical modes are chosen one can proceed as in Sect. 2.2. In particular one can introduce (gauge invariant) creation and annihilation operators by Eq. (2.23), require decomposition (2.24), and finally define local field operators (2.25) in the given gauge.

In the rest of the book this procedure will be implied but not actually used. One just needs spectra of physical modes to calculate corresponding spectral functions, see Chap. 7, and show that calculation of physical quantities does not depend on the choice of the gauge conditions. "Quantization in physical modes" can be related to standard methods and attributes of quantum gauge theories, such as the Faddeev-Popov quantization etc., which are more convenient in interacting theories. We shall briefly comment on this in Sect. 7.8. More intuition on gauge models can be acquired from Exercises 2.6, 2.9, 2.10.

## 2.4 Canonical Quantization

In quantum mechanics one imposes canonical commutation relations

$$[q,\pi]=i\hbar \tag{2.29}$$

between the canonical coordinates $q$ and their respective momenta $\pi$. The general scheme of quantization of free fields introduced above is equivalent to canonical quantization. The canonical momenta $\pi$ are defined by the variational derivative

$$\pi(x)=\frac{\delta\mathscr{L}}{\delta\dot{\varphi}(x)} \tag{2.30}$$

of the Lagrangian $\mathscr{L}$. Here $\dot{\varphi}$ denotes the time derivative and, therefore, the definition of the momenta depends on the choice of the coordinate system. If $t$ is a time coordinate the Lagrangian in this system is defined as the density of the classical action, $I=\int dt\,\mathscr{L}$.

Let us demonstrate equivalence of the two quantization procedures for the scalar field model (1.68) in Minkowski space-time. For a complex scalar field there are two sets of canonical coordinates and conjugate momenta, $\varphi,\pi=\dot{\varphi}^+$ and $\varphi^+,\pi^+=\dot{\varphi}$. Let us fix an inertial frame of reference with the coordinates $x^\mu=(t,\mathbf{x})$ and choose $\Sigma$ as a constant time hypersurface $t=const$. On $\Sigma$

$$\langle f_1,f_2\rangle=i(f_1,\dot{f}_2)-i(\dot{f}_1,f_2), \tag{2.31}$$

where $(f_1,f_2)$ is an inner product in the Hilbert space $\mathrm{L}^2$ on $\Sigma$

$$(f_1,f_2)\equiv\int d^dx\, f_1^*(\mathbf{x})f_2(\mathbf{x}). \tag{2.32}$$

From (2.31) one gets

$$\varphi(f_k)=i(f_k,\dot{\varphi})-i(\dot{f}_k,\varphi)=i(f_k,\pi^+)-i(\dot{f}_k,\varphi). \tag{2.33}$$

It should be emphasized that $f_k(t,\mathbf{x})$ and $\dot{f}_k(t,\mathbf{x})$ at $t$ fixed represent independent variables, the Cauchy data for the solutions $f_k(x)$. If one chooses $f_1=\dot{f}_2=0$, Eqs. (2.18) and (2.33) imply the commutation rules

$$[(\dot{f}_1,\varphi),(f_2^*,\pi)]=i\hbar(\dot{f}_1,f_2)\quad\text{and}\quad[\varphi(\mathbf{x}),\pi(\mathbf{y})]=i\hbar\delta(\mathbf{x}-\mathbf{y}), \tag{2.34}$$

which coincide with (2.29). In the same way one gets other commutators between canonical variables.

In Minkowski space it is easy to construct normalized “modes” $f_i^{(\pm)}$. One of such examples is the so-called plane waves

$$f_{\mathbf{p}}^{(+)}(x)=\frac{1}{\sqrt{2\omega_{\mathbf{p}}}(2\pi)^{d/2}}e^{-i\omega_{\mathbf{p}}t+i\mathbf{px}}, \tag{2.35}$$

$f_{\mathbf{p}}^{(-)}(x)=(f_{\mathbf{p}}^{(+)}(x))^*$. The vector $\mathbf{p}\in\mathbb{R}^d$ is the momentum of the mode, $\omega_{\mathbf{p}}=(\mathbf{p}^2+m^2)^{1/2}$ is the energy. It can be checked, that (2.35) are properly normalized,

$$\langle f_{\mathbf{p}}^{(\pm)},f_{\mathbf{k}}^{(\pm)}\rangle=\pm\delta(\mathbf{p}-\mathbf{k}). \tag{2.36}$$

The operators

$$a^+(\mathbf{p}) = -\langle f_{\mathbf{p}}^{(-)}, \varphi^+\rangle, \qquad b^+(\mathbf{p}) = -\langle f_{\mathbf{p}}^{(-)}, \varphi\rangle \tag{2.37}$$

are creation operators for particles and anti-particles with fixed energies and momenta.

In the solutions (2.35), a transition from one inertial frame to another generates a covariant transformation of quantities $(\omega_{\mathbf{p}}, \mathbf{p})$ as components of a four vector $p^\mu$. This means that (2.35) is a universal set of modes (plane waves) for all inertial observers. This also implies that for all such observers the vacuum state $|0\rangle$ is unique.

## 2.5 Quantum Theory on Stationary Backgrounds

Suppose that external classical background fields are stationary, i.e., there is a coordinate system $x^\mu = (t, x^i)$ where the background fields do not depend on the time coordinate $t$. In this case the energy of an isolated system is conserved. There are two definitions of the energy which can be found in the literature. One is determined in terms of the stress-energy tensor (1.22),

$$E = \int_\Sigma T_{\mu\nu} t^\mu \, d\Sigma^\nu, \tag{2.38}$$

where $t^\mu$ is the Killing vector field which generates translations along the time coordinate $t$. The integral is taken over a space-like surface $\Sigma$ (which can be chosen as a surface of constant time). Another definition of the energy is known as the *canonical energy* or the *Hamiltonian*,

$$H = \sum_i \int d^d x \, \dot{\varphi}_i \frac{\delta L}{\delta \dot{\varphi}_i} - L, \tag{2.39}$$

where $L$ is the Lagrangian of the system and $\varphi_i$ is a set of dynamical variables together with its time derivatives $\dot{\varphi}_i = \partial_t \varphi_i$. Definition (2.39) implies that the Lagrangian does not contain time derivatives higher than the first order. It can be shown [122] that $E$ and $H$ differ by a surface term which vanishes under the suitable boundary conditions, see an example in Exercise 2.11.

If the background is stationary the classical canonical energy $H[f]$ computed for a solution $f$ to the equation of motion (2.1) can be represented as

$$H[f] = \frac{i}{2}(\langle f, \dot{f}\rangle + \langle f^+, \dot{f}^+\rangle). \tag{2.40}$$

For theories with a real (Hermitian) fields one finds

$$H[f] = \frac{i}{2}\langle f, \dot{f}\rangle. \tag{2.41}$$

We leave the proof of these statements in different models for Exercise 2.8.

An important property of the theory on a stationary background is that the time variable is separated. As a consequence, one can introduce a special set of solutions to (2.1) which are the eigenfunctions of the operator $i\partial_t$,

$$i\partial_t f_i^{(\pm)}(x) = \pm\omega_i^{(\pm)} f_i^{(\pm)}(x). \tag{2.42}$$

We assume that $\omega_i^{(\pm)} > 0$. Thus, "+" and "−" modes are eigenfunctions of $i\partial_t$ with positive or negative eigenvalues, respectively. The numbers $\omega_i^{(\pm)}$ determine the spectrum of single-particle excitations and are called the single-particle energies.

The spectrum of single-particle energies is determined by an eigenvalue problem which follows from (2.1). For integer spin fields the operator $P(\partial_t, \partial_k)$ is a second order partial differential operator. For these fields (2.1) is reduced to

$$(P_0\omega^2 + P_1\omega + P_2) f_\omega(x^k) = 0, \tag{2.43}$$

where $P_k$ is a $k$-th order differential operator. For spin 1/2 fields the problem like (2.43) is obtained by taking the square of the Dirac equation (1.74). The operators $P_k$ do not commute between each other in general. Equation (2.43) is a non-linear spectral problem which is discussed in Chap. 6.

The normalization constant in the relativistic product is chosen such that the energies of elementary field excitations (described by $f_i^{(\pm)}$) coincide with frequencies of the modes. To see this for complex fields, we first use (2.40) and (2.42) to get

$$H[f_i^{(\pm)}] = \pm\omega_i^{(\pm)}\langle f_i^{(\pm)}, f_i^{(\pm)}\rangle. \tag{2.44}$$

Then the cases of Bose and Fermi statistics are considered separately.

**Bose Statistics** If the normalization condition (2.20) is satisfied, Eq. (2.44) yields

$$H[f_i^{(\pm)}] = \omega_i^{(\pm)}. \tag{2.45}$$

This equation implies that $H[f] \geq 0$, which may not be the case in general. In static space-times (when the Killing field $t^\mu$ is orthogonal to constant time hypersurfaces) one can guarantee positivity of $H$ for systems whose stress-energy tensor satisfies the so-called *weak energy condition* [156]. The condition requires that $T_{\mu\nu}u^\mu u^\nu \geq 0$ for any time-like vector $u^\mu$.

The energy operator is constructed from its classical analog $H$ when classical fields are replaced with corresponding operators. Substitution of (2.25) in (2.40) and using commutation relations (2.26) yields the quantum Hamiltonian in the following form:

$$H = \sum_i \omega_i^{(+)} a_i^+ a_i + \sum_j \omega_j^{(-)} b_j^+ b_j + E_0. \tag{2.46}$$

The constant $E_0$ in the r.h.s. of (2.46) is given by an infinite series

$$E_0 = \frac{\hbar}{2}\sum_i \omega_i^{(+)} + \frac{\hbar}{2}\sum_j \omega_j^{(-)}. \tag{2.47}$$

The result, as expected, is equivalent to the energy of an infinite number of harmonic oscillators. In field theory, the series (2.47) diverge and require a regularization (a cutoff) at large frequencies $\omega_i^{(\pm)}$. One finds with the help of (2.27) that the ground state is the eigenvector of the energy operator, $H|0\rangle = E_0|0\rangle$. For this reason $E_0$ is called the energy of zero-point fluctuations or the vacuum energy. The vacuum energy will be a special subject of Chap. 9.

Hermitian Bose fields are considered in the same way. In this case there is a single sort of creation and annihilation operators, say $a_i^+$, $a_i$ and the single type of frequencies, $\omega_i = \omega_i^+ = \omega_i^-$. Therefore,

$$H = \sum_i \omega_i a_i^+ a_i + E_0, \tag{2.48}$$

$$E_0 = \frac{\hbar}{2} \sum_i \omega_i. \tag{2.49}$$

To get (2.48) one has to use Eq. (2.41) for the energy.

**Fermi Statistics** If the normalization condition (2.21) is satisfied, it follows from (2.44) that

$$H[f_i^{(\pm)}] = \pm\omega_i^{(\pm)}. \tag{2.50}$$

Thus, the classical energy is negative for modes with negative frequencies. On the quantum level contributions of negative and positive frequency modes to the energy have equal forms and signs because of Fermi statistics. When one uses anticommutation relations (2.26) the energy operator looks as follows:

$$H = \sum_i \omega_i^{(+)} a_i^+ a_i + \sum_j \omega_j^{(-)} b_j^+ b_j + E_0, \tag{2.51}$$

$$E_0 = -\frac{\hbar}{2} \sum_i \omega_i^{(+)} - \frac{\hbar}{2} \sum_j \omega_j^{(-)}. \tag{2.52}$$

The negative constant $E_0$ is the vacuum energy.

**Relation to Classical Mechanics** We finish this section with the following comment. The relativistic product (2.9) is a structure which appears already in the classical mechanics for a finite number of degrees of freedom. Consider a system of $N$ variables $q_k(t)$ whose evolution is described by the Hamilton equations. One can find the corresponding canonical momenta $p_k(t)$ and define the following symplectic form [224]:

$$\Omega(q_1, p_1; q_2, p_2) = i \sum_{k=1}^{N} (q_{1,k}(t) p_{2,k}(t) - p_{1,k}(t) q_{2,k}(t)), \tag{2.53}$$

where $(q_{i,k}, p_{i,k})$ are solutions to the Hamilton equations for the given system. One can show that $\partial_t \Omega(q_1, p_1; q_2, p_2) = 0$ and check that the canonical energy computed on a solution $q_k(t)$ can be written as [224]

$$H[q] = i\Omega(q, \partial_t q). \tag{2.54}$$

Thus, (2.53) is an analog of product (2.9), while (2.54) is an analog of (2.40).

## 2.6 Green's Functions

In this section we introduce a number of the so-called two-point Green's functions. Consider as an example a scalar field $\varphi$ in the Minkowski space-time. The equation of motion is, see (1.69),

$$(-\partial_\mu \partial^\mu + m^2)\varphi(x) = 0. \tag{2.55}$$

With the help of local field operators (2.25) one can define the following functions:

$$G^+(x, x') = \langle 0|\varphi(x)\varphi^+(x')|0\rangle, \tag{2.56}$$

$$G^-(x, x') = \langle 0|\varphi^+(x')\varphi(x)|0\rangle, \tag{2.57}$$

$$iG(x, x') = [\varphi(x), \varphi^+(x')] = G^+(x, x') - G^-(x, x'), \tag{2.58}$$

$$G^{(1)}(x, x') = \langle 0|\varphi(x)\varphi^+(x') + \varphi^+(x')\varphi(x)|0\rangle = G^+(x, x') + G^-(x, x'), \tag{2.59}$$

$$iG_F(x, x') = \theta(t - t')G^+(x, x') + \theta(t' - t)G^-(x, x'), \tag{2.60}$$

$$G_R(x, x') = -\theta(t - t')G(x, x'), \tag{2.61}$$

$$G_A(x, x') = \theta(t' - t)G(x, x'). \tag{2.62}$$

Here $\theta(x)$ is a step function, $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. The names of the functions are the following: $G^+$ and $G^-$ are the Wightman functions, $G$ is the Pauli-Jordan function, $G^{(1)}$ is the Hadamard function, $G_F$ is the Feynman function (or the Feynman propagator), $G_R$, $G_A$ are retarded and advanced Green's functions, respectively.

Since the field operators obey (2.55) the Green's functions are solutions to similar homogeneous or inhomogeneous equations. For instance, it follows from (2.58) that

$$(-\partial_\mu \partial^\mu + m^2)G(x, x') = 0, \tag{2.63}$$

where the differential operator acts either on the argument $x$ or $x'$. The same equation holds for $G^-$, $G^+$, and $G^{(1)}$. For the Feynman function one finds

$$(-\partial_\mu \partial^\mu + m^2)G_F(x, x') = \delta^{(n)}(x - x'), \tag{2.64}$$

where $\delta^{(n)}(x - x') = \delta(t - t')\delta(\mathbf{x} - \mathbf{x}')$. To get the r.h.s. of (2.64) one has to take into account canonical commutation relation (2.34), see Exercise 2.12.

Equations (2.25), (2.26) can be used to rewrite the Green's function in terms of single-particle modes. For instance, for the Wightman and the Pauli-Jordan functions one gets

$$G^+(x, x') = \langle 0|\varphi(x)\varphi^+(x')|0\rangle = \sum_i f_i^{(+)}(x)(f_i^{(+)}(x'))^*, \tag{2.65}$$

$$G^-(x, x') = \langle 0|\varphi^+(x')\varphi(x)|0\rangle = \sum_j (f_j^{(-)}(x'))^* f_j^{(-)}(x), \tag{2.66}$$

$$iG(x, x') = \sum_i f_i^{(+)}(x)(f_i^{(+)}(x'))^* - \sum_j (f_j^{(-)}(x'))^* f_j^{(-)}(x). \tag{2.67}$$

Other Green's functions can be expressed similarly.

The same representations, (2.65)–(2.67), hold for theories in arbitrary background fields. The Green's functions (except for the Pauli-Jordan function which is determined by the commutator) depend on the choice of the vacuum state.

By using (2.65)–(2.67) one can show that the Green's functions above have singularities on the light cone $(x-x')^{\mu}(x-x')_{\mu}=0$. There may be singularities of different types: power or logarithmic singularities, delta-function-like singularities or discontinuities.

It is instructive to give explicit expressions for Green's functions for a massless scalar field in a four-dimensional Minkowski space-time. The modes are defined by (2.35) with $d=3$ and $m=0$. Due to translation invariance of the Minkowski space-time the Green's functions depend on the difference of the arguments, $G(x,x')=G(0,x'-x)\equiv G(x'-x)$. A straightforward computation yields for the Wightman functions, see Exercise 2.16,

$$G^{\pm}(x)=\frac{1}{4\pi^2 s^2}\pm\frac{i}{4\pi}\varepsilon(t)\delta(s^2), \tag{2.68}$$

where $x=(t,\mathbf{x})$, $s^2=s^2(x)\equiv -t^2+\mathbf{x}^2$ is the invariant interval between $x$ and 0, and $\varepsilon(t)=\theta(t)-\theta(-t)$ is the sign function. With the help of (2.58), (2.59), (2.60), and (2.68) one gets the following expressions for the Pauli-Jordan, the Hadamard, and the Feynman functions:

$$G(x)=\frac{1}{2\pi}\varepsilon(t)\delta(s^2), \tag{2.69}$$

$$G^{(1)}(x)=\frac{1}{2\pi^2 s^2}, \tag{2.70}$$

$$G_F(x)=-\frac{i}{4\pi^2 s^2}-\frac{1}{4\pi}\delta(s^2). \tag{2.71}$$

The Pauli-Jordan function (2.69) vanishes outside $s=0$. For massive fields it vanishes under weaker conditions, if the interval is space-like, $s^2(x)>0$. This means that the field operators in causally disconnected points commute. Such a property holds in general, see Exercise 2.18.

## 2.7 Computation of Averages

The two-point Green's functions play an important role in physical applications. They are used in perturbation methods in quantum theories of interacting fields, see discussion in Sect. 7.7. Here we describe how the Green's functions can be used to find expectation values of operators corresponding to physical observables.

As an example, consider the vacuum expectation value for the stress-energy tensor of the scalar field discussed in the previous section. The classical stress-energy tensor in this model is, see (1.70),

$$T_{\mu\nu}=2\partial_{\mu}\varphi^*\partial_{\nu}\varphi-\eta_{\mu\nu}(\partial_{\sigma}\varphi^*\partial^{\sigma}\varphi+m^2\varphi^*\varphi). \tag{2.72}$$

In quantum theory the stress-energy tensor becomes an operator which is obtained from (2.72) by replacing classical fields with the corresponding operators. The vacuum average $\langle 0|T_{\mu\nu}|0\rangle$ suffers from divergences which appear in the averages of products of field at coinciding points, like in $\langle 0|\varphi(x)\varphi(x)|0\rangle$. Such averages are related to the Wightman function (2.56) which, as we have seen already, is singular when its arguments coincide.

To deal with the divergences one uses the so-called point-splitting method. For example, the regularized average of the stress-energy tensor can be defined as

$$\langle 0|T_{\mu\nu}(x)|0\rangle \equiv \lim_{x'\to x} \langle 0|2\partial_\mu\varphi^+(x')\partial_\nu\varphi(x) - \eta_{\mu\nu}(\partial_\sigma\varphi^+(x')\partial^\sigma\varphi(x) + m^2\varphi^+(x')\varphi(x))|0\rangle. \tag{2.73}$$

Here $x$ and $x'$ are close points, such that $x - x'$ is not light-like. This expression can be also written in terms of the Wightman function

$$\langle 0|T_{\mu\nu}(x)|0\rangle = \lim_{x'\to x} [2\partial'_\mu\partial_\nu - \eta_{\mu\nu}(\partial'_\sigma\partial^\sigma + m^2)]G^-(x', x), \tag{2.74}$$

where $\partial_\mu = \partial/\partial x^\mu$ and $\partial'_\mu = \partial/\partial(x')^\mu$. In the limit $x' = x$ the singularities of the Wightman function result in singularities of the average (2.73). The singular terms, however, can be separated from the finite ones and subtracted. The physical justification for this operation, which is called a renormalization, is explained in Sect. 7.5. The example of computation based on formula (2.74) is given in Exercise 9.1 to Chap. 9.

For non-interacting fields the point-splitting method is quite general. Consider a local classical quantity $\mathscr{O}$ which is, like the stress-energy tensor or gauge currents, a quadratic polynomial of the field variables $\varphi$ and its derivatives up to the second order. It can be written as a coincidence limit

$$\mathscr{O}(x) = \lim_{x\to x'} D_{AB}(x, x')\varphi^A(x)\varphi^B(x'), \tag{2.75}$$

where $D_{AB}(x, x')$ is a bi-differential operator, $A$ and $B$ are field indices. In quantum theory the average value of the observable $\mathscr{O}$ is determined by using (2.75)

$$\langle\mathscr{O}(x)\rangle = \lim_{x\to x'} D_{AB}(x, x')\langle\varphi^A(x)\varphi^B(x')\rangle, \tag{2.76}$$

where, as before, the correlator $\langle\varphi^A(x)\varphi^B(x')\rangle$ can be expressed in terms of a two-point Green's function. The physical quantity is obtained from (2.75) after subtracting the divergent parts. The operator $D_{AB}(x, x')$ corresponding to a given $\mathscr{O}$ may be non-unique. It is not a problem if different definitions after subtracting divergences yield the same result.

There is an alternative method of computing the averages of operators which is based on using the effective action and is our main interest. We shall return to this issue in Chap. 7.

## 2.8 Quasinormal Modes

In constructing a quantum theory along the lines of previous sections one may encounter solutions to wave equations (2.1) which look similar to the single-particle modes but have nothing to do with quantum excitations. One type of such modes has a vanishing norm. This may happen because the modes have zero frequency or they are related to pure gauge degrees of freedom, see Exercise 2.6.

In this section we describe another type of classical solutions, the so-called quasinormal modes. Although these modes have complex frequencies and are not normalizable they carry important information about physical properties of the system. As an example we consider a two-dimensional scalar field model with the wave equation

$$(\partial_t^2 - \partial_x^2 + V(x))\varphi(t,x) = 0. \tag{2.77}$$

It is assumed that $-\infty < x < \infty$ and the "background field" is described by a "potential" $V(x)$. We suppose that $V(x)$ is a smooth bounded function with a compact support, such that $V(x) = 0$ if $|x| > b > 0$.

The spectrum of single-particle energies related to eigenvalues of the operator $-\partial_x^2 + V$ has a continuous part, and it is the only part if $V(x) > 0$. How can complex frequency modes appear in this problem? Suppose that a solution to (2.77) is determined at some initial moment, say at $t = 0$, by the Cauchy data, $\varphi(0,x)$, $\partial_t\varphi(0,x)$, which have a compact support. The quasinormal modes appear when one studies asymptotic of $\varphi$ at late times.

Let us start with construction of a general solution to (2.77). We use the Laplace transform

$$\chi(\lambda,x) = \int_0^\infty e^{-\lambda t}\varphi(t,x)\,dt, \tag{2.78}$$

which enables us to represent the solution in the integral form

$$\varphi(t,x) = \frac{1}{2\pi i}\int_C e^{t\lambda}\chi(\lambda,x)\,d\lambda. \tag{2.79}$$

The contour $C$ in the complex plane goes parallel to the imaginary axis such that $\Re\,\lambda = a > 0$. The function $\chi(\lambda,x)$ is defined through a one-dimensional problem

$$(\lambda^2 - \partial_x^2 + V(x))\chi(\lambda,x) = j(\lambda,x) \tag{2.80}$$

with a "source" determined by the Cauchy data,

$$j(\lambda,x) \equiv \partial_t\varphi(0,x) + \lambda\varphi(0,x). \tag{2.81}$$

To get (2.80) from (2.77) one has to start with the Laplace transform of $\partial_t^2\varphi$ and integrate by parts

$$\int_0^\infty e^{-\lambda t}\partial_t^2\varphi(t,x)\,dt = \lambda^2\chi(\lambda,x) - j(\lambda,x). \tag{2.82}$$

The solution to (2.80) can be written with the help of a Green's function $G_\lambda(x, y)$

$$\chi(\lambda, x) = \int_{-\infty}^{\infty} dy\, G_\lambda(x, y) j(\lambda, y), \tag{2.83}$$

$$(\lambda^2 - \partial_x^2 + V(x))G_\lambda(x, y) = \delta(x - y). \tag{2.84}$$

The differential operator in (2.84) does not depend on time. We shall describe a method how to construct $G_\lambda(x, y)$ in a way which differs from the procedure applicable to time-dependent Green's functions (2.65)–(2.67). Consider the homogeneous equation

$$(\lambda^2 - \partial_x^2 + V(x)) f(\lambda, x) = 0. \tag{2.85}$$

A pair of independent solutions to (2.85), $f_k$, can be determined by their asymptotic behavior at $x \to \pm\infty$. Because $V(x) = 0$ for $|x| > b$ one can choose the following asymptotics

$$f_1(\lambda, x) \sim e^{-\lambda x}, \quad x \gg b, \tag{2.86}$$

$$f_2(\lambda, x) \sim e^{+\lambda x}, \quad x \ll -b. \tag{2.87}$$

The Wronskian of the system

$$W(\lambda) = \frac{1}{2}(f_2'(\lambda, x) f_1(\lambda, x) - f_1'(\lambda, x) f_2(\lambda, x)), \tag{2.88}$$

does not depend on the coordinate $x$ and is not vanishing on independent solutions, $W(\lambda) \neq 0$. One can check (see Exercise 2.21) that a solution to (2.84) can be written as

$$\begin{gathered} G_\lambda(x, y) = \frac{g_\lambda(x, y)}{W(\lambda)}, \\ g_\lambda(x, y) = \theta(x - y) f_1(\lambda, x) f_2(\lambda, y) + \theta(y - x) f_1(\lambda, y) f_2(\lambda, x). \end{gathered} \tag{2.89}$$

It can be shown that the Laplace transform $\chi(\lambda, x)$, see (2.78), is uniquely determined for $\Re\,\lambda > 0$ by (2.83), (2.89) and is a bounded function provided that the Cauchy data (the "source" $j(\lambda, x)$) have a compact support.

The way how one can determine the late time behavior of the solution $\varphi(t, x)$ provided that its initial perturbation is localized in a finite region is the following. Consider representation (2.79) and use (2.83), (2.88) to get

$$\varphi(t, x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dy \int_C d\lambda\, e^{t\lambda} \frac{g_\lambda(x, y)}{W(\lambda)} j(\lambda, y). \tag{2.90}$$

Because $t > 0$ one can add to the contour $C$ a semicircle lying in left half of the complex plane. Then integration in (2.90) is equivalent to integration over a closed contour and can be performed by using the Cauchy theorem. If $W(\lambda)$ have complex zeros $\lambda_k$ ($\Re\,\lambda_k < 0$) the contour integral in (2.90) acquires contributions from the residues of $1/W(\lambda)$ at $\lambda_k$. At late $t$ the main contribution to (2.90) is determined by the pole $\lambda_0$ with the smallest real part $|\Re\,\lambda_0|$. Therefore, at late $t$

$$\varphi(t, x) \simeq e^{t\lambda_0} \varphi_0(x), \tag{2.91}$$

where $\varphi_0(x)$ is some function.

It can be shown that for positive potentials $V(x)$ with a compact support the Wronskian always has a countable number of zeros. The idea behind finding these zeros is quite simple. If $W(\lambda) = 0$ the corresponding solutions, $f_k(\lambda, x)$, are not independent. Suppose that for a certain complex value $\lambda$ ($\Re\,\lambda > 0$) the wave equation (2.77) allows for a solution, $\bar{f}(\lambda, x)$, which has both asymptotics, $\bar{f}(\lambda, x) \sim e^{\pm\lambda x}$ at $x \to \mp\infty$. It then follows from (2.86), (2.87) that $\lambda$ is one of the zeros of the Wronskian.

Such solutions $\bar{f}(\lambda, x)$ are called the quasinormal modes. The complex numbers $\lambda$ which are the zeros of the Wronskian are called the quasinormal spectrum. The ringing frequencies of a bell, which can be heard, are related to the quasinormal spectrum. The inverse of $|\Re\,\lambda_0|$ yields the lifetime of the main overtone which decays the last.

Let us emphasize once again that quasinormal frequencies despite their physical importance are not eigenvalues of the operator $-\partial_x^2 + V$ because the corresponding modes are not normalizable.

## 2.9 Literature Remarks

Commutation relation (2.18) of free fields on a gravitational background has been used by a number of authors, see e.g. a pioneering paper by Chernikov and Tagirov on quantum theory in de Sitter space-time [65]. An alternative way would be to postulate, by following DeWitt [77], the local commutation relations as

$$[\varphi(x), \varphi^+(x')] = iG(x, x'),$$

where $G(x, x')$ is the Pauli-Jordan function which can be defined in classical theory by Eq. (2.67). This scheme of quantization is manifestly covariant and is equivalent to (2.18) on globally hyperbolic space-times.

A quantization procedure of fields of different spins in Minkowski space-time along with properties and singularity structure of Green's functions is described in detail in the classical book by Bogoliubov and Shirkov [40]. Among the modern monographs on quantum field theory we mention the book by Peskin and Schroeder [205] and the book by Weinberg [253].

Quantization of gauge theories and constrained dynamics is presented in many books, see e.g. [108, 137]. We should note that a method of "quantization in physical modes" discussed in Sect. 2.3 may fail on some curved backgrounds, see e.g. [105, 242]. Although what is described in Sect. 2.3 and later in Sect. 7.8 is enough to demonstrate applications of the spectral theory to different problems with quantum gauge fields.

Quasinormal modes encode important characteristics of frequencies and lifetimes of gravitational waves emitted at late stages by a black hole after its perturbation, see more on this subject in the book by S. Chandrasekhar [62]. A recent review of quasinormal modes of stars and black holes can be found in [171, 173]. Possible role of quasinormal modes in quantum gravity theory is discussed in [160].

There are several reasons why we do not discuss in this book higher-spin fields (spins 3/2, 2 and etc.). The main reason is that we study here the Lagrangian field theories while a problem of a Lagrangian formulation for higher spin fields is open in general and details in its resolution are still missing. Classical free Lagrangian higher spin field theories in Minkowski space-time were formulated in the middle of 70th of the last century. However, coupling of these fields to arbitrary external backgrounds or interaction among higher spin fields faces the problem of consistency. A consistent interacting massless spin 2 field theory is general relativity, non-contradictory interacting massless spin 3/2 and 2 fields enter in supergravity. At present, there exists Lagrangian formulation for massless and massive arbitrary higher spin fields in anti de Sitter (AdS) space-time. Besides, spin 3/2 and 2 field Lagrangian formulation exists in the Einstein space. As for general higher spin interaction, it seems that Lagrangian formulation should include an infinite tower of all higher spin fields (like in string theory) and it is not so clear how to quantize such theories. The higher spin fields in the AdS space can be quantized by standard methods and, in principle, the mathematical techniques which are considered in this book can be applied to study an effective action in this case. As far as we know, such a consideration has never been carried out in general, besides spin 3/2 and 2 fields.

Recommended Exercises are 2.5, 2.6, 2.8, 2.10, 2.16.

## 2.10 Exercises

**Exercise 2.1** Prove that the integral $Q=\int_\Sigma d\Sigma^\mu\, j_\mu$ on a hypersurface $\Sigma$ does not change under smooth local transformations $\Sigma$ if $\nabla^\mu j_\mu = 0$.

**Exercise 2.2** Consider the following linear combination of single-particle modes $f_i^{(\pm)}$:

$$\tilde{f}_i^{(+)} = \sum_k \alpha_{ik}^{(+)} f_k^{(+)} + \sum_p \beta_{ip}^{(+)} f_p^{(-)}, \tag{2.92}$$

$$\tilde{f}_j^{(-)} = \sum_k \alpha_{jk}^{(-)} f_k^{(+)} + \sum_p \beta_{jp}^{(-)} f_p^{(-)}, \tag{2.93}$$

where $\alpha_{ik}^{(\pm)}$ and $\beta_{ip}^{(\pm)}$ are some complex numbers.

1) Find relations between $\alpha_{ik}^{(\pm)}$ and $\beta_{ip}^{(\pm)}$ which guarantee that $\tilde{f}_i^{(\pm)}$ form another set of single-particle modes which satisfy (2.19)–(2.21).
2) Find a transformation from creation and annihilation operators determined by modes $f_i^{(\pm)}$ to creation and annihilation operators determined by modes $\tilde{f}_i^{(\pm)}$ (this transformation is called the Bogoliubov transformation after N.N. Bogoliubov who introduced it in the theories of superfluidity and superconductivity).
3) Calculate the number of particles of the new sort in the vacuum state (2.27).

**Exercise 2.3** Consider free scalar field model (1.69) in a general gravitational and gauge background. Prove that the general quantization scheme presented in Sect. 2.1 coincides with the canonical quantization.

**Exercise 2.4** Consider a theory of free quantum fields on a globally hyperbolic space-time $\mathscr{M}$. Prove that the quantization condition (2.18) implies that

$$[\varphi(x_1), \varphi^+(x_2)]_\pm = 0 \tag{2.94}$$

when points $x_1$ and $x_2$ are on a Cauchy surface $\Sigma$. By the definition of $\Sigma$ (see Sect. 1.6) such points are casually independent.

**Exercise 2.5** Consider a vector field action

$$I[A, g] = -\frac{1}{2}\int d^n x\,\sqrt{-g}(\nabla_\nu A_\mu \nabla^\nu A^\mu + R_{\mu\nu} A^\mu A^\nu + M^2 A_\mu A^\mu), \tag{2.95}$$

where $R_{\mu\nu}$ is the Ricci-tensor of the background metric. What is the difference between this model and model (1.75)? Why quantization of (2.95) yields a theory with unphysical properties?

**Exercise 2.6** What is the difference between the massive and massless vector models (1.75)? Note that the massless model is the Maxwell theory in a vacuum. Identify physical degrees of freedom in the Maxwell theory.

**Exercise 2.7** Prove that the norm $\langle\psi, \psi\rangle$ of $c$-number valued (non-Grassmann) spinor fields defined on a smooth space-like hypersurface by (2.9), (2.11) is positive. Prove the following property:

$$\langle\psi_1^c, \psi_2^c\rangle = \langle\psi_2, \psi_1\rangle, \tag{2.96}$$

where $\psi^c$ denotes a charge conjugated spinor, see (1.66). If $\psi_1$ and $\psi_2$ are Grassmann fields, a minus sign appears in the equation above.

**Exercise 2.8** Derive formulae (2.40), (2.41) for the canonical energy in stationary backgrounds for models of scalar (1.68), spinor (1.73), vector (1.75), and non-Abelian gauge fields (1.79).

**Exercise 2.9** Let $\omega_i$ be the spectrum of single-particle energies in a field model on a stationary background. One can define a spectral function

$$\Phi = \sum_i f(\omega_i), \tag{2.97}$$

where $f(x)$ is some smooth function which decays fast enough to ensure convergence of the series. One of the examples of the spectral function is the regularized vacuum energy, see (2.47), where $f(x) = x/2$ ($f(x) = 0$ for $x > a$ where $a$ is a regularization parameter). Other examples are studied below.

Find a relation between the spectral functions of the vector models (1.75) and (2.95). Consider both massive and massless cases.

**Exercise 2.10** Consider a theory of linear order perturbations $A_\mu$ of a $SU(N)$ gauge field over the background field $B_\mu$, where $B_\mu$ is a static solution to Yang-Mills equations (1.78). The equations for the perturbations are (1.79).

By analogy with the Maxwell theory, see Exercise 2.6, study the single-particle spectrum $\omega_i$ of the perturbations in the Lorentz-like gauge $[D^\mu, A_\mu] = 0$. Find a representation of the spectral function (2.97) in this gauge in terms of the spectral functions of some unconstrained fields.

**Exercise 2.11** Consider a model of a real scalar field with the so-called non-minimal coupling between the field and the curvature scalar

$$I = -\frac{1}{2}\int d^n x\,\sqrt{g}(\partial^\mu\varphi\partial_\mu\varphi + m^2\varphi^2 + \xi R\varphi^2). \tag{2.98}$$

By using definitions (2.38), (2.39) calculate the energy and the Hamiltonian for this model on a static background. Find the difference between the two quantities and show that it is reduced to a surface term.

**Exercise 2.12** Prove the equation

$$(-\partial_\mu\partial^\mu + m^2)G_F(x, x') = \delta^{(n)}(x - x')$$

for the scalar Feynman function in Minkowski spacetime, see Sect. 2.6.

**Exercise 2.13** Consider a massless field on a circle, i.e. field in two-dimensional space-time which obeys the periodic condition in spatial coordinate, $\varphi(t, x + l) = \varphi(t, x)$. Prove that the Wightman function for this model has the form

$$G^+(0, x^\mu) = -\frac{1}{4\pi}\left[\ln\left(-4\sin\frac{au}{2}\sin\frac{av}{2}\right) + \frac{i}{2}a(u + v)\right], \tag{2.99}$$

where $a = 2\pi/l$, $x^\mu = (t, x)$, $u = t - x$, $v = t + x$ and $\Im t = \epsilon > 0$.

**Exercise 2.14** Get the following expression for the Wightman function of the massless two-dimensional field on an interval of the length $l$:

$$G^+(x^\mu, (x')^\mu) = \frac{1}{4\pi}\ln\left[\frac{\cos a\Delta t - \cos a(x + x')}{\cos a\Delta t - \cos a(x - x')}\right], \tag{2.100}$$

where $a = \pi/l$ and $\Delta t = t' - t$, $\Im t' > 0$. The boundary condition for the field is $\varphi(t, l) = \varphi(t, 0) = 0$.

**Exercise 2.15** By using explicit expressions for the Wightman functions (2.99), (2.100) derive canonical commutation relation for the massless scalar field on a circle and on an interval.

**Exercise 2.16** Prove expressions (2.68) for the Wightman functions of a massless scalar field in four-dimensional Minkowski space-time.

**Exercise 2.17** Consider a massive scalar field in four-dimensional Minkowski space-time. Show that the Feynman, advanced and retarded Green's functions can be defined in the so-called momentum representation

$$\mathscr{G}(0,x) = \frac{1}{(2\pi)^4}\int d^4p\, \frac{e^{-ipx}}{p^2+m^2}, \tag{2.101}$$

where each of these functions is specified by a prescription how to pass the poles in the denominator. The following notations are used in (2.101): $p$ is a four-dimensional momentum, $p = (p_0, \mathbf{p})$, $px = -p_0 t + \mathbf{px}$, $p^2 = -p_0^2 + \mathbf{p}^2$.

Find also analogous prescription for the Wightman and Pauli-Jordan functions.

**Exercise 2.18** By using results of Exercise 2.17 demonstrate the Lorentz invariance of Green's functions.

**Exercise 2.19** Consider the Cauchy problem

$$(-\partial_\mu \partial^\mu + m^2)\varphi(x) = 0, \tag{2.102}$$

$$\varphi(x)|_{t=0} = \varphi_1(\mathbf{x}), \qquad \dot{\varphi}(x)|_{t=0} = \varphi_2(\mathbf{x}). \tag{2.103}$$

Show that a solution to (2.102), (2.103) can be written with the help of the Pauli-Jordan function,

$$\varphi(x) = \int d\mathbf{y}[\partial_{t_y} G(x,y)|_{t_y=0}\varphi_1(\mathbf{y}) - G(x,y)|_{t_y=0}\varphi_2(\mathbf{y})]. \tag{2.104}$$

**Exercise 2.20** By using the point-splitting method, see (2.76), define the average value of the electric current for scalar and spinor field models (1.68), (1.73) in Minkowski spacetime. For the classical current use the definition (1.71).

**Exercise 2.21** Consider Eq. (2.84) for a one-dimensional Green's function $G_\lambda(x,y)$. Let $f_k(\lambda, x)$ be two independent solutions to the homogeneous equation (2.85) and $W(\lambda)$ be their Wronskian (2.88). Demonstrate that the Green's function can be written as

$$G(x,y) = \frac{1}{W(\lambda)}(\theta(x-y) f_1(\lambda,x) f_2(\lambda,y) + \theta(x-y) f_1(\lambda,y) f_2(\lambda,x)). \tag{2.105}$$

# Part II
# Spectral Geometry

# Chapter 3
# Operators and Their Spectra

## 3.1 Differential Operators on Manifolds

To develop the quantum theory further one needs a number of mathematical notions. We begin with the theory of linear operators, the mathematical cornerstone of the quantum theory. The form of the operators is related to classical field equations, as discussed in Chaps. 1 and 2. We start with a bit abstract setting and find out which properties of the operators are essential to have well-defined spectral problems so that the eigenmodes can be used in the quantization. We are not going to present the mathematical theory of operators on manifolds to any degree of completeness. Our aim is to introduce the main notions so that the reader may consult more advanced literature. For the reasons which will be explained latter we assume in the most part of this and next Chapters that the base manifold is Riemannian (the Euclidean signature). Lorentzian manifolds are the subject of separate digressions.

In physical applications one mostly deals with the operators of a Laplace type. In a local basis such operators can be represented as

$$L = -(g^{\mu\nu}(x)\partial_\mu\partial_\nu + a^\mu(x)\partial_\mu + b(x)), \tag{3.1}$$

where $g^{\mu\nu}$ is the Riemannian metric, $a^\mu(x)$ and $b(x)$ are some matrix valued functions. In general, instead of $g^{\mu\nu}$ there can be a matrix valued function. We shall not consider this case because it leads to more complicated spectral properties. Note also that operators having a matrix valued coefficient in front of the second derivative term are called non-minimal. One can transform $L$ to an explicitly covariant form

$$L = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + E), \tag{3.2}$$

where the covariant derivative $\nabla = \nabla^{[R]} + \omega$ contains both Riemann, $\nabla^{[R]}$, and "gauge", $\omega$, parts. One may express

$$\begin{aligned} \omega_\rho &= \frac{1}{2} g_{\nu\rho}\big(a^\nu + g^{\mu\sigma}\Gamma^\nu_{\mu\sigma}\big), \\ E &= b - g^{\nu\mu}\big(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu - \omega_\sigma\Gamma^\sigma_{\nu\mu}\big), \end{aligned} \tag{3.3}$$

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where $\Gamma^{\sigma}_{\nu\mu}$ is the Christoffel symbol (1.4) associated with $g_{\mu\nu}$. These formulae can be used to extend our definition of the Laplace operator from a local chart to sections of a vector bundle, i.e., to make the operator globally defined. This is done simply by saying that $\omega$ is a connection, and $E$ is an endomorphism on $\mathscr{F}(\mathscr{M})$. In the present context, this means that we have fixed transformation properties of $\omega_\mu$ and $E$ under gauge transformation from the structure group of the bundle. Namely, $\omega$ is transformed as in Eq. (1.40), and $E$ is transformed homogeneously, $E \to \mathbf{g}E\mathbf{g}^{-1}$. The simplest example of the operator (3.1) is the scalar Laplacian

$$\Delta\varphi \equiv -\nabla^{\mu}\nabla_{\mu}\varphi = -g^{-1/2}\partial_{\mu}\big(g^{1/2}g^{\mu\nu}\partial_{\nu}\varphi\big). \tag{3.4}$$

This operator acts on scalar functions. Another example is an operator acting on vector fields,

$$\Delta^{(1)}V_{\mu} \equiv \big(-\nabla^{\alpha}\nabla_{\alpha}\delta^{\nu}_{\mu} + R^{\nu}_{\mu}\big)V_{\nu}. \tag{3.5}$$

The form of $\Delta^{(1)}$ is dictated by classical equations for vectors, see, e.g. Exercises 2.5, 2.6.

An important class of partial differential operators is given by operators of the Dirac type. By definition, $\not{D}$ is of the Dirac type if its square, $\not{D}^2$, is of the Laplace type. Clearly, any operator of the form

$$\not{D} = i\gamma^{\mu}\partial_{\mu} + V(x) \tag{3.6}$$

with an arbitrary zero order part $V(x)$ is of the Dirac type. It is convenient to rewrite (3.6) in explicitly covariant form,

$$\not{D} = i\gamma^{\mu}(\partial_{\mu} + \omega^{[s]}_{\mu}) + \tilde{V}(x), \tag{3.7}$$

where $\omega^{[s]}_{\mu}$ is the spin-connection, see Eq. (1.57). An example of (3.7) appeared in the equation of motion (1.74) for a spin $1/2$ field $\psi$. The associated Laplace operator is

$$\Delta^{(1/2)}\psi \equiv (i\gamma^{\mu}\nabla_{\mu})^2\psi = \left(-\nabla^{\alpha}\nabla_{\alpha} + \frac{1}{4}R\right)\psi, \tag{3.8}$$

where $R$ is the scalar curvature. This equation is called the Lichnerowicz formula.

Generalizing the examples given above one arrives at the notion of a differential operator. In a local basis a differential operator of order $p$ reads

$$\sum_{k=0}^{p} c_k^{\mu_1\ldots\mu_k}(x)\partial_{\mu_1}\ldots\partial_{\mu_k}. \tag{3.9}$$

It is essential that operator (3.9) includes partial derivatives up to some finite order.

Differential operators are defined on some spaces of functions. Let us discuss briefly what are these spaces. Since the operator $L$ in (3.1) contains two partial derivatives, the functions it acts on should have at least two well behaving derivatives. These functions belong to the so-called Sobolev space. As we know from Chap. 2, to construct a quantum theory one needs the space of square integrable functions, denoted by $\mathrm{L}^2$. Let us remind, that any convergent sequence in $\mathrm{L}^2$ has its limit also in $L^2$. Thus the space is complete and is a Hilbert space. We shall suppose that our operators can be extended to $L^2$ in a meaningful way. For example, this can be done with the help of the Fourier transform because the action of $L$ and $\not{D}$ on the plane waves is well defined. The action of these operators on the square integrable functions can be obtained by summing up the Fourier series.

The Fourier transformation can be used to introduce even more general notion of the operators, the pseudodifferential operators. Let $f(x)$ be a function on $\mathbb{R}^n$ and let $\tilde{f}(k)$ be its Fourier component,

$$f(x) = \int e^{ikx}\,\tilde{f}(k)\,d^n k. \tag{3.10}$$

The action of a pseudodifferential operator $P$ on $f$ is defined as

$$Pf(x) = \int e^{ikx}\,p(x,k)\tilde{f}(k)\,d^n k, \tag{3.11}$$

where $p(x,k)$ is a smooth function of both arguments. If $p(x,k)$ grows at large $|k|$ as $|k|^q$, one says that $P$ has the order $q$. Usually one also requires that $l$th derivative of $p(x,k)$ with respect to $k$ grows no faster as $|k|^{q-l}$. The function $p(x,k)$ is called the symbol of $P$. In this way one can define operators of a fractional order. One can also extend the notion of a pseudodifferential operator to operators on arbitrary manifolds.

Differential operators belong to the class of pseudodifferential operators. For example, the symbol of a Laplace operator $L$ reads

$$p_L(x,k) = g^{\mu\nu}(x)k_\mu k_\nu - i a^\mu(x)k_\mu - b(x). \tag{3.12}$$

Consider an operator $P$ of a finite positive order $q$. One defines the leading symbol $\sigma_P(x,k)$ as the part of $p(x,k)$ which scales as $|k|^q$ at large $k$. The importance of the leading symbol for studying differential operators will become evident soon.

Another important notion is the ellipticity property of differential operators. By definition, $P$ is elliptic if its leading symbol $\sigma_P(x,k)$ is non-degenerate for all $x \in \mathscr{M}$ and all $k \neq 0$. Since the leading symbol is understood as a linear map on the fiber space the ellipticity means that the map is an isomorphism.

For $P = L$ the leading symbol is $\sigma_L = k^2 \times I$, where $I$ is the identity map in the fiber space, i.e. it is an identity matrix with internal or spin indexes, see for example (3.5). Consequently, the Laplace operator $L$ is elliptic. For the Dirac type operators $\sigma_{\not{D}} = -k^\mu \gamma_\mu I_{\text{int}}$, where $I_{\text{int}}$ is an identity matrix in the internal (or gauge) space. Obviously, $\sigma_{\not{D}}^2 = k^2 I_{\text{int}} \otimes I_{\text{spin}}$ is non-degenerate for $k \neq 0$. Hence $\sigma_{\not{D}}$ is non-degenerate itself, and Dirac operators are elliptic.

Ellipticity means that at large momenta the term with the highest derivatives dominates, so that the spectrum behaves in a predictable way. An elliptic operator on a compact manifold may have only a finite number of zero eigenvalues. For a Laplacian even a stronger statement is possible: the number of negative eigenvalues on a compact manifold is finite.

In most of the physical applications one deals with Hermitian vector bundles, see Sect. 1.5. This means that for any two elements, $f_1$, $f_2$ from the fiber at each point of the base manifold $\mathscr{M}$ there is a bilinear form $(f_1(x), f_2(x))_x$ called the fiber metric. For example, for tangent vectors, this may be just their scalar product, $(V_1(x), V_2(x))_x = g_{\mu\nu}(x)V_1^\mu(x)V_1^\nu(x)$. The bilinear form is assumed to be Hermitian, $(f_1(x), f_2(x))_x = (f_2(x), f_1(x))_x^*$, and invariant under an action of the structure group. By using the fiber metric one can define the inner product of two elements from the corresponding Hilbert space $\mathrm{L}^2$,

$$(f_1, f_2) = \int_{\mathscr{M}} d^n x \sqrt{g}\,(f_1(x), f_2(x))_x. \tag{3.13}$$

The inner product (3.13) should not be confused with the relativistic product (2.9) which we used for the quantization. The inner product is used to define the norm $\|\cdot\|$ on the Hilbert space. By definition, $\|f\| = (f, f)^{1/2}$.

By using (3.13) one can introduce an operator $D^\dagger$ adjoint to an operator $D$,

$$(D^\dagger f_1, f_2) = (f_1, Df_2). \tag{3.14}$$

If $D^\dagger = D$ one says that $D$ is symmetric. To check whether a (pseudo-)differential operator is symmetric one has to integrate by parts in the inner product (3.14) to move $D$ from $f_1$ to $f_2$. Usually this can be done for closed manifolds. If the base manifold has a boundary the boundary conditions on $f_1$ and $f_2$ have to ensure that $D$ is symmetric. We shall return to this problem in Sect. 3.2.

A symmetric operator is called selfadjoint if the domains in $\mathrm{L}^2$ where $D$ and $D^\dagger$ are defined coincide. We shall mostly ignore this requirement assuming that operators are selfadjoint on a suitable domain.

Selfadjoint operators have real eigenvalues. A Laplacian $L$ is selfadjoint if in a suitable local basis the connection $\omega_\mu$ is represented by an anti-Hermitian matrix, and if $E(x)$ is Hermitian in the same basis (if the bundle metric $(,)_x$ is assumed to be positive definite).

We now consider several simple examples. The first example is the scalar Laplace operator

$$\Delta = -\partial_1^2 - \partial_2^2 - \cdots - \partial_n^2 \tag{3.15}$$

on an $n$-dimensional torus, $\mathscr{M} = T^n$. By definition, functions on $T^n$ satisfy the periodic conditions $f(x^1, \ldots, x^\mu, \ldots, x^n) = f(x^1, \ldots, x^\mu + l^\mu, \ldots, x^n)$ for each coordinate $x^\mu$. The periods $l^\mu$ are real numbers. One requires that $\Delta$ acts on the periodic functions which belong to the corresponding Hilbert space $\mathrm{L}^2(T^n)$. The eigenspectrum of $\Delta$ can be easily found by taking into account the periodicity conditions,

$$\Delta f_k(x) = \lambda_k f_k(x), \qquad f_k = (l^1 l^2 \ldots l^n)^{-1/2} \exp(ik_\mu x^\mu), \tag{3.16}$$

$$\lambda_k = k_\mu k^\mu, \qquad k_\mu = 2\pi q^\mu / l^\mu, \tag{3.17}$$

where $\{q^\mu\} \in \mathbb{Z}^n$. The functions $f_k$ form an orthonormal set in $\mathrm{L}^2$,

$$(f_k, f_{k'}) = \int_{\mathscr{M}} d^n x\, f_k^*(x) f_{k'}(x) = \delta_{k,k'}. \tag{3.18}$$

Decomposition of functions from $L^2(T^n)$ in the basis $\{f_k\}$ is known as the Fourier series.

Another example is the scalar Laplacian on a unit two-sphere, see (1.97). By using (3.4) in the coordinates $(\theta, \tau)$ on $S^2$ one obtains the following expression:

$$\Delta f = -\left(\partial_\theta^2 + \cot\theta\,\partial_\theta + (\sin\theta)^{-2}\partial_\tau^2\right) f. \tag{3.19}$$

In quantum mechanics this operator is known as the square of angular momentum. Its eigenfunctions are the spherical harmonics $Y_{l,m}$,

$$\Delta Y_{l,m}(\theta,\phi) = l(l+1)Y_{l,m}(\theta,\phi), \tag{3.20}$$

where $l = 0, 1, 2, \dots$, and $m$ changes between $-l$ and $+l$. The eigenvalues of $\Delta$ depend on $l$ only. Therefore, the degeneracy of each eigenvalue is $d_l = 2l + 1$.

In general, the eigenvalues and degeneracies of the scalar Laplacian on a unit $n$-sphere $S^n$ are

$$\lambda_l = l(l+n-1), \quad l = 0, 1, 2, \dots, \tag{3.21}$$

$$d_l = \frac{(2l+n-1)(l+n-2)!}{l!(n-1)!}. \tag{3.22}$$

Exact spectrum on $S^n$ can be also found for non-zero spin Laplacians.

## 3.2 Boundary Conditions

In physical applications one has to deal with classical and quantum problems on manifolds with boundaries, see Sect. 1.8. If the manifold $\mathscr{M}$ has a boundary, $\partial\mathscr{M}$, the definition of a differential operator $D$ requires certain conditions on $\partial\mathscr{M}$ for functions which $D$ acts on. The need for boundary conditions is known from early years of mathematical physics. Indeed, the equation

$$Df = J \tag{3.23}$$

with a given $J$ cannot be solved for $f$ unless one imposes suitable restrictions on the behavior of $f$ on $\partial\mathscr{M}$.

Let us briefly discuss the number of boundary conditions. Suppose that the manifold $\mathscr{M}$ has the topology $\partial\mathscr{M} \times \mathbb{R}^1$. For an initial value (Cauchy) problem for the equation $Df = J$ where $D$ is a differential operator of order $q$ and $\partial\mathscr{M}$ is an "initial" surface one requires to fix $q$ independent initial data, for example, $f$ and its first $q - 1$ normal derivatives on $\partial\mathscr{M}$. Implicitly, in this problem there is another boundary, which is the "final" surface. In case of two explicit boundaries, $\partial\mathscr{M}_1$ and $\partial\mathscr{M}_2$, one can as well impose $q/2$ conditions at each boundary, that is $q/2$ conditions along the whole boundary $\partial\mathscr{M}_1 \cup \partial\mathscr{M}_2$. The same is, of course, true when $\mathscr{M}$ has a topology of a ball and a closed boundary $\partial\mathscr{M}$. One can conclude that for an order $q$ operator $q/2$ boundary data on $\partial\mathscr{M}$ are required. For Laplace type operators this means one boundary condition for each field component. For a Dirac operators one has to fix one half of the spinor components.

The spectral problems following from the boundary conditions must be mathematically consistent. The rest of this section will be devoted to formulation and study of such consistency conditions.

Let us give some examples of the boundary problems mentioned above. The first example is the one-dimensional Laplacian

$$L = -\partial_x^2 \tag{3.24}$$

on the interval $[0, l]$. To fix the spectrum one has to impose a single condition at each of the points $x = 0$ and $x = l$. There are two most common choices how to do this. One can either fix the values of the field,

$$f|_{x=0,l} = 0, \tag{3.25}$$

or the values of first derivatives,

$$\partial_x f|_{x=0,l} = 0. \tag{3.26}$$

These are the so-called Dirichlet, (3.25), and Neumann, (3.26), boundary conditions. The eigenfunctions of $L$ in both cases are easy to find:

$$\begin{aligned} &\text{Dirichlet:} \quad f_k = l^{-1/2} \sin(kx\pi/l), \quad k = 1, 2, \ldots; \\ &\text{Neumann:} \quad f_k = l^{-1/2} \cos(kx\pi/l), \quad k = 0, 1, 2, \ldots. \end{aligned} \tag{3.27}$$

The eigenvalues read

$$\lambda_k = \left(\frac{k\pi}{l}\right)^2. \tag{3.28}$$

These eigenvalues are real and non-negative. Indeed, after integrating by parts and using (3.25) or (3.26) one gets

$$(Lf_1, f_2) - (f_1, Lf_2) = -\partial_x f_1 \cdot f_2\big|_{x=0}^{x=l} + f_1 \cdot \partial_x f_2\big|_{x=0}^{x=l} = 0. \tag{3.29}$$

Therefore, both Dirichlet and Neumann boundary conditions make the operator (3.24) selfadjoint on the interval on a suitable domain of definition. One may notice now that the asymptotic behavior of the spectrum for large $k$ for operators on an interval and on a circle, see (3.17), are very similar.

The next case is the Laplace operator on a manifold with a closed boundary. The simplest example is a two-dimensional disc (1.96). The Laplace operator reads

$$Lf = -\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\tau^2\right) f, \tag{3.30}$$

where we used coordinates $0 \le \tau < 2\pi$, $0 \le \rho \le \rho_0$ with $\rho_0$ being the radius of the disc. Consider the Dirichlet boundary conditions

$$f(\rho_0, \tau) = 0. \tag{3.31}$$

One can separate the radial and angular dependence of $f$ and write the eigenmodes as

$$f_{k,\lambda}(\rho, \tau) = J_{|k|}(\rho\lambda^{1/2}) \exp(ik\tau), \tag{3.32}$$

where $k \in \mathbb{Z}$, $J_{|k|}$ are the Bessel functions, and $\lambda$'s are eigenvalues,

$$Lf_{k,\lambda} = \lambda f_{k,\lambda}, \tag{3.33}$$

defined by the boundary condition

$$J_{|k|}(\rho_0 \lambda^{1/2}) = 0. \tag{3.34}$$

The situation when the spectrum can be found only implicitly, through the solutions of a transcendental equation, is typical for boundary value problems. From the theory of the Bessel functions it follows that the eigenvalues are real and positive. For large $z$ and positive integer $k$ we have the following asymptotic expression for the Bessel functions

$$J_k(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{k\pi}{2} - \frac{\pi}{4}\right). \tag{3.35}$$

Therefore, large eigenvalues are defined by zeros of the cosine in (3.35). Again, like in the previous examples, one may notice that the spectrum looks very similar to spectrum on a torus, see (3.17).

We now give an example when the presence of boundaries does lead to essential modifications of the spectrum and physical properties of the system (the example is borrowed from Appendix B of [176]). Consider the Laplace operator

$$L = -\partial_1^2 - \partial_2^2 \tag{3.36}$$

on a cylinder $\mathscr{M} = [0, 1] \times S^1$ $(0 \le x^2 \le 2\pi)$ with the boundary conditions

$$\partial_1 f|_{x^1=0} = 0, \qquad (\partial_1 + i\alpha\partial_2) f|_{x^1=1} = 0. \tag{3.37}$$

There are two sets of eigenmodes,

$$f_{k_1 k_2} = \exp(ik_2 x^2) \cos(k_1 x^1), \tag{3.38}$$

$$\bar{f}_{\bar{k}_1 k_2} = \exp(ik_2 x^2) \cosh(\bar{k}_1 x^1), \tag{3.39}$$

where $k_2 = 0, \pm 1, \pm 2, \dots$. The both sets satisfy (3.37) at $x^1 = 0$ while the condition at $x^1 = 1$ determines discrete values of $k_1$ and $\bar{k}_1$,

$$k_1 \tan k_1 = -\alpha k_2, \qquad \bar{k}_1 \tanh(\bar{k}_1) = \alpha k_2. \tag{3.40}$$

Obviously, the eigenvalues of the first set (3.38) are $k_1^2 + k_2^2$ and $L$ is non-negative. The eigenvalues of the second set are $-\bar{k}_1^2 + k_2^2$. For positive $\alpha k_2$ the second relation (3.40) always has two solutions. For sufficiently large $|k_2|$ they are

$$\bar{k}_1 \approx \pm \alpha k_2 \tag{3.41}$$

up to exponentially small corrections. Thus, the eigenvalues $-\bar{k}_1^2 + k_2^2$ of $L$ are positive if $|\alpha| < 1$ and negative if $|\alpha| > 1$. That is for $|\alpha| > 1$ the operator $L$ has infinitely many "negative" modes.

The example above is not of a pure academic interest, similar boundary conditions appear in the theory of open strings (cf. Chap. 10). In that case $|\alpha| = 1$ corresponds to a critical value of background gauge fields.

An infinite number of "negative" modes may be related to peculiar properties of a system. If $L$ is the operator which defines eigenfrequencies of fluctuations (see Chap. 2) through $L\varphi = \omega^2\varphi$, negative eigenvalues of $L$ correspond to the states with imaginary frequencies, which, in turn, indicates instability of the system.

The boundary conditions should lead to problems with physically reasonable properties. To achieve this, it is not enough to require that the operator is elliptic, like for closed manifolds. One has to make sure that the boundary value problem satisfies the so-called *weak* and *strong ellipticity conditions*. We do not discuss these conditions here. The interested reader can consult Literature Remarks at the end of this Chapter for further remarks and references. Instead, we list several boundary conditions which are known to lead to elliptic boundary value problems for the Laplace operator (3.2).

For this purpose it is convenient to introduce a boundary operator $\mathscr{B}$ and write the boundary conditions as

$$\mathscr{B}f = 0. \tag{3.42}$$

Since we shall mostly discuss free field theory, we consider linear operators $\mathscr{B}$. We also do not include inhomogeneous terms in the right hand side of (3.42).

**Dirichlet Boundary Conditions** are the simplest boundary conditions which generalize (3.25). The boundary operator in this case reads

$$\mathscr{B}f \equiv \mathscr{B}_D f = f|_{\partial\mathscr{M}}, \tag{3.43}$$

so that $\mathscr{B}_D$ simply restricts $f$ to the boundary.

**Robin Boundary Conditions**, which are also called generalized Neumann boundary conditions, correspond to the boundary operator

$$\mathscr{B}_N f = (\nabla_n + \mathscr{S}) f|_{\partial\mathscr{M}} \tag{3.44}$$

with $\mathscr{S}$ being a matrix valued function on $\partial\mathscr{M}$. In the mathematical language $\mathscr{S}$ is an endomorphism of the restriction of the vector bundle $\mathscr{E}$ to the boundary. There are two advantages of the Robin boundary conditions over the Neumann ones. First of all, the partial derivative $\partial_n$ is not a covariant object. Therefore, it is natural to write down boundary conditions with the covariant derivative $\nabla_n$. Adding $\mathscr{S}$ is very important for technical reasons: one can vary connection and $\mathscr{S}$ simultaneously so that the boundary operator (3.44) remains invariant.

**Mixed Boundary Conditions** One can also mix up the boundary conditions defined above by requiring that some components of $f$ satisfy Dirichlet, and the rest satisfies Robin boundary conditions. More precisely, one introduces two complementary projectors, $\Pi_D$ and $\Pi_N$, $\Pi_D + \Pi_N = 1$, and defines the boundary operator as

$$\mathscr{B}_{\rm mix} f = \Pi_D f|_{\partial\mathscr{M}} + (\nabla_n + \mathscr{S})\Pi_N f|_{\partial\mathscr{M}}. \tag{3.45}$$

As we shall see below, natural boundary conditions for spin 1/2 and spin 1 fields belong to this class.

**Oblique boundary conditions** generalize the condition (3.37). The operator has the form

$$\mathscr{B}_{\mathrm{o}} = (\nabla_n + \Gamma^i \tilde{\nabla}_i + \mathscr{S}) f|_{\partial\mathscr{M}}, \tag{3.46}$$

where $\tilde{\nabla}_i$ are covariant derivatives with respect to the metric on the boundary, see Sect. 1.8. These conditions lead to an elliptic boundary value problem only if absolute values of all eigenvalues of all matrices $\Gamma^i$ do not exceed one.

For all these boundary conditions the spectrum of the Laplace operator behaves “nicely”. In particular, there could be at most a finite number of negative and zero modes. Also, the distribution of large positive eigenvalues is universal: it is governed by the Weyl formula (5.39) to be discussed in Sect. 5.4.

Let us now discuss in more detail boundary conditions for the Dirac operator. As we have already explained above, since the Dirac operator is a first order differential operator one needs boundary conditions on one half of the spinor components. Let us choose a projector $\Pi_D$ which selects a half of the components and impose the Dirichlet boundary conditions

$$\Pi_D \psi|_{\partial\mathscr{M}} = 0. \tag{3.47}$$

There may be many different choices for $\Pi_D$. It is natural to require that the normal component of the spinor current vanishes on the boundary,

$$\psi^\dagger \gamma^n \psi|_{\partial\mathscr{M}} = 0, \tag{3.48}$$

where $\gamma^n = \gamma^\mu n_\mu$ and $n_\mu$ is a unit vector orthogonal to the boundary $\partial\mathscr{M}$. Suppose that $\mathscr{M}$ is an even-dimensional manifold. Then it is easy to show that for the projector

$$\Pi_D = \frac{1}{2}(1 \pm i\gamma_* \gamma^n), \tag{3.49}$$

where $\gamma_*$ is the chirality matrix, the condition (3.48) is indeed satisfied.

The Dirac operator squared is a Laplace type operator. Therefore, the eigenvalue problems for $\not{D}$ and for the corresponding Laplacian $\not{D}^2$ should be in some sense equivalent. However, for $L = \not{D}^2$ one has to double the number of boundary conditions. To see where the missing conditions come from consider the eigenvalue equation $\not{D}\psi = \lambda\psi$. By acting with the projector $\Pi_D$ on both sides of this equation one obtains the condition

$$\Pi_D \not{D}\psi|_{\partial\mathscr{M}} = 0, \tag{3.50}$$

which should hold at least on eigenfunctions of $\not{D}$. This second condition leads to mixed boundary conditions for $\not{D}^2$. Indeed, consider a simple case when $\mathscr{M} = \mathbb{R}^{n-1} \times \mathbb{R}_+$ and $\not{D} = i\gamma^\mu \partial_\mu$. Then

$$\Pi_D \not{D}\psi = i\gamma^n \partial_n (1 - \Pi_D)\psi + i\gamma^j \partial_j \Pi_D \psi, \tag{3.51}$$

where $\partial_j$ denote partial derivatives with respect to the coordinates tangential to the boundary. The second term on the right hand side of (3.51) vanishes on the boundary due to (3.47). The first term together with (3.50) yields Neumann boundary conditions for $(1 - \Pi_D)\psi$. Hence, we arrived at mixed boundary conditions of the type (3.45).

## 3.3 Bounded and Compact Operators

Let us now discuss the properties of infinite-dimensional operators related to boundedness of their spectra.

Let $P$ be a linear operator on a Hilbert space with the norm $\|\cdot\|$. $P$ is called *bounded* if for $f$ from its domain

$$\sup_{\|f\|\leq 1} \|Pf\| < \infty. \tag{3.52}$$

To understand whether an operator is bounded, one should check how it acts on the Fourier harmonics. Let us note that if $f = \sum f_k c_k$, where $f_k$ are normalized Fourier harmonics, then $\|f\| = (\sum |c_k|^2)^{1/2}$. Then, the set $\|f\| \leq 1$ looks like a unit ball in the space spanned by $c_k$. If the operator is selfadjoint its boundedness is equivalent to the boundedness of the set of the eigenvalues. As an example, let us consider the standard Laplacian $L$ (3.24) on an interval. The spectrum is given by (3.28), and it is unbounded, so that the operator (3.24) is unbounded as well. All eigenvalues of the operator

$$Q = (l^2 L + 1)^{-1} \tag{3.53}$$

are between 0 and 1 and, therefore, the operator $Q$ is bounded.

For further purposes we should distinguish bounded and compact domains. An example of a bounded domain is a "unit ball", i.e. a set of functions defined by $\|f\| \leq 1$. A compact space is defined as a space where any infinite sequence of elements contains a convergent subsequence. The space of functions satisfying $\|f\| \leq 1$ is not compact in the infinite-dimensional case. As a consequence of the definition, bounded operators map a bounded domain to a bounded domain. By using the stronger requirement one can also consider *compact* operators which map a bounded domain to a compact domain. To see whether a linear operator is compact it is enough to check this property on the unit ball. If $e_k$ is a (countable) basis of normalized elements in the Hilbert space, the sequence with the $n$-th element given by $\frac{1}{2}e_n$ is contained in the unit ball, but does not contain a convergent subsequence since the distance between any two elements is $1/\sqrt{2}$. To make the image of the ball compact, the operator $P$ must provide enough "squashing" of the ball, i.e., enough regularization for high momenta. The operator of multiplication by a constant $c$, while being obviously bounded, does not give such "squashing" (consider, e.g., the sequence $ce_n$ which belongs to the image of unit ball). Therefore, the operator of multiplication by a constant is not compact. The operator $Q$ defined above in

Eq. (3.53) is compact. For example, the sequence $\{f_n\}$ with $f_l$ being an eigenfunction of $L$, see Eq. (3.27), is mapped to a convergent sequence. To prove this, it is enough to note that the distance $\|Qf_k - Qf_n\| \le 2(N^2\pi^2+1)^{-1}$ for $k, n > N$, and that it goes to zero as $N \to \infty$. Similarly, one can prove that the image of *any* sequence contained in the unit ball has a convergent subsequence. Compact operators are norm limits of sequences of finite-rank operators.

It is convenient to study the properties of operators by looking at their spectrum. However, not every operator has a complete set of eigenfunctions. For an operator $T$ consider $T^\dagger T$, which is at least formally selfadjoint and should have a complete set of non-negative real eigenvalues. By taking positive square roots of these eigenvalues, one can define the "absolute value" of $T$ as $|T| = (T\dagger T)^{1/2}$. The eigenvalues of $|T|$ are called singular values of $T$ and denoted $s_k(T)$. For a compact operator $T$ one can arrange $s_k(T)$ in a non-increasing order. If $p$ is a non-negative real number, a compact operator $T$ is said to be Schatten $p$-class if the sum $\sum_k s_k(T)^p$ is convergent. One writes then $T \in \mathscr{L}^p$. Two important particular cases are the Hilbert-Schmidt class $\mathscr{L}^2$ and the trace class $\mathscr{L}^1$. For $T \in \mathscr{L}^1$ trace of $T$ is absolutely convergent and does not depend on the orthogonal basis used.

In the infinite-dimensional case many "nice" properties of the operators hold modulo a compact operator. For example, any Laplacian is invertible modulo the projector on its zero subspace. In a similar sense one can define complex powers of Laplace type operators [227].

The notions of bounded, unbounded and compact operators will be used when discussing the spectral triples in Sect. 11.5.

## 3.4 Lorentzian Signature

Let us now discuss operators on Lorentzian manifolds, see Sect. 1.2. A typical example of this kind is the D'Alambert operator in the Minkowski space

$$\Box = -\eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_0^2 - \partial_1^2 - \cdots - \partial_{n-1}^2. \tag{3.54}$$

The eigenfunctions of this operator are plane waves (2.35) which can be written (up to a normalization coefficient) as

$$\Box e^{ikx} = k^2 e^{ikx} = (-k_0^2 + k_1^2 + \cdots + k_{n-1}^2)e^{ikx}, \tag{3.55}$$

where $kx = k_\mu x^\mu$ and $k_\mu$ is a "momentum" of the plane wave. The spectrum of $\Box$ is obviously not positive definite. The operators whose leading symbol behaves similar to (3.55) are called second order *hyperbolic* operators, and this notion replaces the notion of the elliptic operators.

Even more drastic changes appear for the Dirac operator. The inner product of the spinor fields which is invariant with respect to the structure group $Spin(1, n-1)$, see Sect. 1.5, contains the $\gamma^0$ matrix,

$$(\psi', \psi) = \int_{\mathscr{M}} \sqrt{-g}\,\bar{\psi}'\psi, \quad \bar{\psi} \equiv i\psi^\dagger\gamma^0. \tag{3.56}$$

This product is not positive definite. The notion of selfadjointness is defined with respect to (3.56). Since the $\gamma$-matrices in Minkowski space are "$\gamma^0$-Hermitian"

$$\gamma^0 \gamma^\mu \gamma^0 = \gamma^{\mu\dagger} \tag{3.57}$$

the hermiticity condition for the Dirac operator reads

$$\gamma^0 \not{D} \gamma^0 = \not{D}^\dagger . \tag{3.58}$$

If there are boundaries, one has to impose boundary conditions on a half of the spinor components. This can be done similarly to the Euclidean case by choosing (3.47). If the normal vector to the boundary is spacelike, $n_\mu n^\mu = 1$, one can take the projector, for example, as

$$\Pi_D = \frac{1}{2}(1 + \gamma^n). \tag{3.59}$$

Then the boundary conditions are

$$\Pi_D \psi|_{\partial\mathcal{M}} = 0, \qquad \bar{\psi}(1 - \Pi_D)|_{\partial\mathcal{M}} = 0 \tag{3.60}$$

and the current flow through the boundary vanishes,

$$\bar{\psi}\gamma^n \psi|_{\partial\mathcal{M}} = 0, \tag{3.61}$$

if both $\bar{\psi}$ and $\psi$ satisfy (3.60). This last property makes it possible to prove hermiticity of the Dirac operator in the presence of boundaries, see Exercise 3.2. Note, that in contrast to the Euclidean case, projector (3.59) does not contain the chirality matrix $\gamma_*$. Therefore, boundary conditions (3.60) exist on both even- and odd-dimensional manifolds.

## 3.5 Literature Remarks

The theory of differential and pseudodifferential operators is a large area of mathematics. To study the basics one can consult any textbook on mathematical physics or on functional analysis, as, e.g. the classics [195, 218] or more recent [141]. A more focused exposition of pseudodifferential operators can be found in Shubin [233]. Boundary value problems are discussed in Grubb [148]. Further references will be given in the next Chapter.

In Sect. 3.2 we have mentioned the weak and strong ellipticity conditions for operators on manifolds with boundaries. The idea behind these conditions which we have not discussed in details, is very simple, though the technical side is rather complicated. One formulates an auxiliary one-dimensional boundary value problem keeping in the operator just the normal derivatives and replacing all tangential derivatives by corresponding momenta. One further simplifies this boundary value problem by picking up the "main" parts of the operator and of the boundary condition, a procedure similar to extraction of the leading symbol of an operator. Then one analyses the spectrum of this simplified boundary value problem. If there are no eigenvalues in some "forbidden" area of the complex plane, the boundary value problem is elliptic. For further reading we suggest the references [133, 148]. A nice simple explanation of the ellipticity properties of boundary value problems is given in [33].

Local boundary conditions for the Dirac operator, called the bag boundary conditions, were introduced in [66, 67]. Spectral theory of these boundary conditions was developed [50], see also [181].

## 3.6 Exercises

**Exercise 3.1** Prove that the operator $L$ in Eq. (3.36) is symmetric under the boundary conditions (3.37). Check by a direct computation that the modes from different sets (3.38) and (3.39) are orthogonal.

**Exercise 3.2** Prove that under boundary condition (3.47) with (3.49) the Dirac operator (3.7) is symmetric (provided that $\tilde{V}(x)$ is hermitian).

**Exercise 3.3** On any two-dimensional manifold a vector field $V_\mu$ can be decomposed as

$$V_\mu = \varrho_{,\mu} + \varepsilon_{\mu\nu}\varphi^{,\nu} + V_\mu^H. \tag{3.62}$$

This decomposition is called the Hodge-de Rham decomposition. In the language of differential forms, the first term on the right hand side corresponds to exact form, the second—to co-exact forms, and the third one is a harmonic vector field satisfying $\Delta^{(1)} V_\mu^H = 0$. Find the spectrum of vector Laplacian (3.5) on the unit $S^2$ by using the decomposition (3.62) and the fact that there no harmonic vectors on the two-sphere.

**Exercise 3.4** Find the spectrum of the Dirac operator on $S^2$. Use eigenfunctions of the scalar Laplacian on $S^2$ and results of Exercise 1.17 to construct explicitly the eigenfunctions of the Dirac operator.

**Exercise 3.5** Prove the following property for the Dirac operator squared (the Lichnerowicz formula):

$$(i\gamma^\mu\nabla_\mu)^2 = -\nabla^\alpha\nabla_\alpha + \frac{1}{4}R,$$

where $R$ is the scalar curvature of the base manifold. This property is used to define the spinor Laplacian $\Delta^{(1/2)}$, see (3.8).

**Exercise 3.6** Consider a Dirac type operator

$$\not{D} = i\gamma^\mu(\partial_\mu + V_\mu + i\gamma_* A_\mu) \tag{3.63}$$

Suppose that the base manifold is flat and bring the operator $\not{D}^2$ to the canonical form $-(\nabla^2 + E)$. For $A_\mu = 0$ the relation which is required to be obtained is called the Weitzenböck formula.

# Chapter 4
# Heat Equation

## 4.1 The Heat Kernel

Let $L$ be an elliptic second-order differential operator acting on sections of a vector bundle over a Riemannian manifold $\mathscr{M}$. Positivity is not assumed, so that a finite number of negative eigenvalues is allowed. We shall actually concentrate on operators of Laplace type, though most of the results will be valid in a more general context. For the given operator $L$ one can define an important object known as the *heat operator* $e^{-tL}$. This can be done with the help of the so-called heat equation:

$$(\partial_t + L)u(x;t) = 0, \quad \text{for } t > 0 \tag{4.1}$$

with the initial condition

$$u(x;0) = f(x), \tag{4.2}$$

where $f(x)$ is a function from $\mathrm{L}^2$. The solution $u(x;t)$ can be written as $u(x;t) = e^{-tL}f(x)$ thus defining the heat operator $e^{-tL}$.

This construction also determines a kernel, the *heat kernel* $K(x,y|t)$,

$$u(x;t) = \int d^n y\, K(x,y|t) f(y). \tag{4.3}$$

The equation for the heat kernel can be written in the form

$$(\partial_t + L_x)K(x,y|t) = 0, \tag{4.4}$$

$$K(x,y|0) = \delta^{(n)}(x,y), \tag{4.5}$$

where $\delta^{(n)}(x,y)$ is the kernel of the unit operator on the space of smooth sections of the vector bundle. If $L$ is a scalar Laplacian on $\mathbb{R}^n$, this kernel is simply the Dirac delta-function $\delta^{(n)}(x-y)$.

As an example, let us consider a free Laplacian $\Delta = -\partial_\mu^2$ on a torus $T^n$ (cf. (3.15)). Let us expand the function $f$ in a Fourier series $f = \sum_k c_k f_k(x)$, where $f_k$ is plane wave (3.16). The heat operator is diagonal in the plane wave basis. It maps $e^{-t\Delta} : c_k \mapsto e^{-tk^2} c_k$. We see, that for $t > 0$ the heat operator improves the behavior

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of the Fourier coefficients $c_k$ at large momenta $k_\mu$ and, consequently, it also makes the functions more smooth. In particularly, for a positive $t$, the heat operator exists and maps $\mathrm{L}^2$ to $C^\infty$. For this reason, $e^{-t\Delta}$ is called an *infinitely smoothing* operator. Obviously, the presence of some lower powers of $k$ in the operator, as it happens in a Laplacian on a curved Riemannian manifold $\mathscr{M}$, does not change our conclusion since $e^{-tk^2}$ dominates over all other contributions anyway. Therefore, *the existence of the heat operator is related only to the property of the symbolic spectrum* (*the ellipticity*). Other conventional assumptions, such as, for instance, the property of self-adjointness of the operator $L$, are not needed for this purpose.

Precisely the same reason, namely the fall-off properties of the heat operator at large momenta, guarantees existence of the *heat trace* on the space of square integrable functions $\mathrm{L}^2$,

$$K(Q, L; t) = \mathrm{Tr}_{\mathrm{L}^2}(Q \exp(-tL)), \tag{4.6}$$

where $Q$ is a partial differential operator. We shall mostly consider the cases when $Q$ is a function (a zero-order operator), or even when $Q$ is the unity. In this latter case we shall use the notation

$$K(L; t) \equiv K(1, L; t). \tag{4.7}$$

This is a spectral function which can be written as

$$K(L; t) = \sum_\lambda e^{-t\lambda}, \tag{4.8}$$

where the sum is taken over all eigenvalues $\lambda$ of $L$. Let us emphasize again that (4.8) does not require that $L$ is a self-adjoint operator.

Suppose $L$ is an elliptic second order partial differential operator on a manifold $\mathscr{M}$ of dimension $n$, where $\mathscr{M}$ either a compact manifold or $\mathscr{M}$ has a boundary and the boundary conditions for $L$ belong to one of the classes of strongly elliptic boundary conditions listed in Sect. 3.2. As will be shown below there is a full asymptotic series for any smooth function $f$ as $t \to +0$

$$K(f, L; t) = \mathrm{Tr}_{\mathrm{L}^2}(f \exp(-tL)) \simeq \sum_{p=0}^\infty t^{\frac{p-n}{2}} a_p(f, L). \tag{4.9}$$

This asymptotic series is called full since the summation over $p$ extends to infinity. In this book we call $a_p$ the *heat kernel coefficients*.

## 4.2 Asymptotics of the Heat Kernel

We start our discussion of the asymptotic behavior of the heat trace with several examples of operators with known spectrum. The heat trace can be represented as an infinite series, and our task will be to evaluate the series at small $t$. A very useful instrument is the Poisson summation formula

$$\sum_{k=-\infty}^{\infty} h(2k\pi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dy\, h(y) e^{-iky}, \tag{4.10}$$

which is valid for any absolutely integrable bounded function $h(y)$. To illustrate how this formula works let us consider an asymptotics of the following simple sum at small $t$:

$$\sum_{k=-\infty}^{\infty} e^{-tk^2}. \tag{4.11}$$

Formula (4.10) can be applied to (4.11) if we choose $h(y) = \exp(-ty^2(2\pi)^{-2})$. The integral over $y$ is easy to calculate,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dy\, h(y) e^{-iky} = \sqrt{\frac{\pi}{t}} e^{-\frac{k^2\pi^2}{t}}. \tag{4.12}$$

This shows that all terms in the sum on the right hand side of (4.10) are exponentially small except for $k = 0$. Therefore, at $t \to +0$,

$$\sum_{k=-\infty}^{\infty} e^{-tk^2} \simeq \sqrt{\frac{\pi}{t}} + \mathscr{O}(e^{-1/t}). \tag{4.13}$$

Now we are able to evaluate the asymptotic expansion of the heat kernel for the Laplacian (3.15) on $T^n$. The spectrum is given by (3.16) and (3.17).

$$K(\Delta; t) = \sum_{q \in \mathbb{Z}^n} \exp\left( -t \sum_{\mu} \frac{2\pi q_\mu^2}{l_\mu^2} \right) \simeq \frac{l_1 l_2 \dots l_n}{(4\pi t)^{n/2}} + \mathscr{O}(e^{-1/t}). \tag{4.14}$$

The symbol $\mathscr{O}(e^{-1/t})$ denotes terms which vanish faster than any power of $t$ and are not relevant for our purposes. One can see that asymptotic expansion (4.14) is indeed of the form announced above in Eq. (4.9).

Another example where the asymptotic expansions can be obtained explicitly is the Laplace operator $\Delta$ on unit spheres. The spectrum in this case is known, see (3.20). For heat kernels on $S^2$ and $S^3$ one finds

$$K(\Delta_{S^2}; t) \simeq \frac{1}{t} + \frac{1}{3} + \frac{t}{15} + \mathscr{O}(t^2), \tag{4.15}$$

$$K(\Delta_{S^3}; t) \simeq \frac{\sqrt{\pi}}{4} \left( \frac{1}{t^{3/2}} + \frac{1}{t^{1/2}} + \frac{t^{1/2}}{2} \right) + \mathscr{O}(e^{-1/t}), \tag{4.16}$$

see Exercises 4.2 and 5.2. These expressions again confirm Eq. (4.9).

## 4.3 DeWitt Approach

The most powerful and the most general method of evaluating the heat trace asymptotics was suggested in 1975 by Gilkey [132]. Many works in physics used an earlier method by DeWitt [77] which is based on recursion relations between the heat kernel coefficients at non-coinciding arguments. We start with a brief explanation of DeWitt's approach.

Let us first consider a free Laplacian on $\mathbb{R}^n$ with unit flat metric. A solution to the heat equation (4.4) with the initial condition (4.5) for $L = -\partial_\mu^2$ reads

$$K(x, y|t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{(x-y)^2}{4t}\right) \tag{4.17}$$

(cf. Exercise 4.1).

By using this result, let us try to guess the heat kernel of a scalar Laplacian on a curved manifold without boundaries in the limit when points $x$ and $y$ are close and the parameter $t$ is small. One expects that in this limit expression (4.17) holds up to curvature corrections. DeWitt proposed the following ansatz

$$K(x, y|t) \sim \frac{1}{(4\pi t)^{n/2}} \Delta_{VVM}^{1/2}(x, y) e^{-\frac{\sigma^2(x,y)}{4t}} \sum_{p=0}^{\infty} b_p(x, y) t^n. \tag{4.18}$$

Here $g(x) = \det g_{\mu\nu}(x)$, $\sigma^2(x, y)$ is the geodesic distance between two close points $x$ and $y$ with coordinates $x^\mu$ and $y^\mu$, respectively. It is assumed that there are no caustics, i.e., geodesic lines form a regular coordinate system near $x$ or $y$. A biscalar determinant

$$\Delta_{VVM}(x, y) = [g(x)g(y)]^{-1/2} \det\left[-\frac{1}{2}\frac{\partial^2\sigma^2(x, y)}{\partial x^\mu \partial y^\nu}\right] \tag{4.19}$$

is called the Van-Vleck–Morette determinant. If $x$ and $y$ are close to each other, the kernel (4.18) looks similar to the flat-space kernel (4.17). The coefficients $b_p$ describe corrections due to the curvature, and the presence of $\Delta_{VVM}(x, y)$ makes the whole expression covariant.

Let us give an idea how to substantiate ansatz (4.18) and find the coefficients $b_p(x, y)$ for the case of the scalar Laplacian $L = -\nabla_\mu \nabla^\nu$. We work in the so-called Riemann normal coordinates (RNC) centered at the point $y$. In RNC, the coordinates of the other point $x$ have the following meaning: $x^\mu = sl^\mu$ where $s$ is the length of the geodesics connecting points $x$ and $y$, while $l^\mu$ is a unit vector at the point $y$ which is tangent to this geodesic curve. If the caustics are absent there is a single geodesic connecting any two points. In addition we require that metric at the point $y$ coincides with $\eta_{\mu\nu}$, a flat metric. Thus, by the definition of RNC $\sigma^2(x, y) = x^\mu x^\nu \eta_{\mu\nu}$. The other advantage of RNC is that the geodesic equation looks as in flat space-time $d^2x^\mu/ds^2 = 0$.

In RNC instead of (4.18) we have

$$K(x, y|t) = \frac{1}{(4\pi t)^{n/2}} \Delta_{VVM}^{1/2}(x) e^{-\frac{x^2}{4t}} \sum_{p=0}^{\infty} b_p(x) t^p, \tag{4.20}$$

where we have taken into account that $\Delta_{VVM}(x, y) = g^{-1/2}(x) \equiv \Delta_{VVM}(x)$. This expansion can be substituted in the heat equation (4.4) to get the relation

$$\sum_{p=0}^{\infty} b_p(x)\left(\left(p-\frac{n}{2}\right)t^{p-1}+\frac{x^2}{4t^2}t^p\right)$$
$$=\Delta_{VVM}^{-1/2}\sum_{p=0}^{\infty}\left(\Delta_{VVM}^{1/2}b_p\left(-\frac{1}{4t}\nabla^2x^2+\frac{(\nabla x^2)^2}{16t^2}\right)\right.$$
$$\left.-\frac{1}{t}x^\mu\nabla_\mu(\Delta_{VVM}^{1/2}b_p)+\nabla^2(\Delta_{VVM}^{1/2}b_p)\right)t^p. \tag{4.21}$$

Next we note that

$$(\nabla x^2)^2=4x^2,$$
$$\nabla^2x^2=\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\nabla_\nu x^2)=\Delta_{VVM}\partial_\mu(\Delta_{VVM}^{-1}2x^\mu)=2x^\mu\Delta_{VVM}\partial_\mu\Delta^{-1}+2n$$

and equate coefficients in front of the powers of $t$ in (4.21) to get the recursion relations

$$(p+1)b_{p+1}+x^\mu\partial_\mu b_{p+1}=\Delta_{VVM}^{-1/2}\nabla^2(\Delta_{VVM}^{1/2}b_p), \tag{4.22}$$

$$x^\mu\partial_\mu b_0=0. \tag{4.23}$$

It remains to show that in the DeWitt ansatz (4.18) there is a solution to the recursion relations. This can be easily shown by finding explicit form of the first coefficients. Because $K(x,y|t)=\delta^{(D)}(x-y)$ in the limit $t\to 0$ we conclude that $b_0=1$. Let us compute now the next coefficient $b_1$ in the limit $x^\mu=0$ (i.e. when the points $x$ and $y$ coincide). By taking into account that $dx^\mu/ds=x^\mu/s$ one comes to equation for the Levi-Civita connection

$$\Gamma^\lambda_{\mu\nu}(x)x^\mu x^\nu=0. \tag{4.24}$$

It can be solved perturbatively when point $x$ is close to $y$. In particular, the form of the metric compatible with (4.24) and the choice of RNC is

$$g_{\mu\nu}(x)=\eta_{\mu\nu}-\frac{1}{3}R_{\mu\alpha\nu\beta}x^\alpha x^\beta+O(x^4). \tag{4.25}$$

From (4.22) we have

$$b_1(0)=\nabla^2\Delta_{VVM}^{1/2}(x)\quad \text{at } x=0.$$

It follows from (4.25) that

$$g(x)=1-\frac{1}{3}R_{\mu\nu}x^\mu x^\nu+O(x^4),\qquad \Delta_{VVM}^{1/2}=1+\frac{1}{12}R_{\mu\nu}x^\mu x^\nu+O(x^4).$$

Therefore, $\nabla^2\Delta_{VVM}^{1/2}=\frac{1}{6}R$ at $x=0$ and

$$b_1=\frac{1}{6}R.$$

This result can be used to find first terms in the asymptotic expansion of the trace of the heat kernel at small $t$

$$K(-\nabla^2;t) = \int d^n x\, g^{1/2} K(x,x|t) \sim \frac{1}{(4\pi t)^{n/2}} \int_{\mathscr{M}} d^n x\, g^{1/2} (b_0 + b_1 t + \cdots), \quad (4.26)$$

valid for a compact manifold without boundaries. By comparing with (4.9) we obtain

$$a_{2p} = (4\pi)^{-n/2} \int_{\mathscr{M}} d^n x\, g^{1/2} b_p. \quad (4.27)$$

In particular,

$$a_0 = (4\pi)^{-n/2} \int_{\mathscr{M}} d^n x\, g^{1/2} = \frac{\mathrm{vol}\,\mathscr{M}}{(4\pi)^{n/2}}, \quad a_2 = (4\pi)^{-n/2} \int_{\mathscr{M}} d^n x\, g^{1/2} \frac{R}{6}. \quad (4.28)$$

We see, that the odd-numbered coefficients $a_{2k+1}$ vanish, which is a general feature of all Laplace type operators on manifolds without boundaries.

## 4.4 Gilkey Approach

This approach consists of two main steps. First, one uses general properties of the heat kernel expansion to fix the heat kernel coefficients up to several unknown constants. Second, one applies some "functorial" relations between the heat kernels of different operators and uses particular case calculations to find these unknown constants. The ultimate success of this method depends on the choice of a proper family ("category") of spectral problems under the consideration. If the family is too wide, the combinatorial complexity becomes overwhelming so that practical calculations are not possible. If the family is too narrow, one cannot find enough useful relations between the members of this family.

In this section we consider generalized Laplacians (3.2) on compact manifolds without boundaries. For such operators there is an asymptotic expansion (4.9). What can we say about the coefficients $a_k$ of this expansion on general grounds? One can prove that all $a_k(f,L)$ are locally computable. This means that they can be expressed as integrals over the manifold $\mathscr{M}$ of local invariants. For an operator (3.1) these local invariants are bundle traces of local covariant expressions linear in $f$ and polynomial in $E$, the gauge field strength $\Omega_{\mu\nu}$ (Eq. (1.41)), the Riemann tensor $R_{\mu\nu\rho\sigma}$ (Eq. (1.10)) and their covariant derivatives. There are two different symmetries which have to be taken into account. The first one is the diffeomorphism invariance which simply tells us that the result for the heat trace asymptotics must not depend on a particular choice of the coordinate system. In practice this means that all vector indices in the polynomials described above should be contracted in pairs. The second invariance is related to gauge transformations. The gauge transformations are defined by the formula (1.40) for $\omega_\mu$ and change $E$ as $\mathbf{g}E\mathbf{g}^{-1}$. The operator $L$ transforms homogeneously, $L \to \mathbf{g}L\mathbf{g}^{-1}$, and all factors of $\mathbf{g}$ are canceled out after taking the trace. All relevant invariants have been already described in Sects. 1.2 and 1.5.

In the rest of this section computation of the heat coefficients will be divided in several steps. Each of the steps serves to describe either a particular property of the heat asymptotics or a useful technique.

**Dimensional Analysis of the Structure of the Heat Coefficients** To restrict powers of the fields appearing in the heat kernel coefficients we have to define canonical mass dimensions of all quantities. We assign the dimension $-1$ to the coordinate, $[\![x^\mu]\!] = -1$, then the dimension of the derivative is $+1$, $[\![\partial_\mu]\!] = 1$. The same dimension should be assigned to the covariant derivative, $[\![\nabla_\mu]\!] = 1$, and $[\![\Omega_{\mu\nu}]\!] = 2$. We keep the metric dimensionless, $[\![g_{\mu\nu}]\!] = 0$. Since $E$ appears in $L$ in a linear combination with $\nabla^2$ we have to put $[\![E]\!] = 2$. Also, $[\![R_{\mu\nu\rho\sigma}]\!] = 2$. The canonical dimension of a monomial is simply a sum of canonical dimensions of multipliers. Since all expressions below will be linear in the smearing function $f$, one can choose any value for its canonical dimension. We take $[\![f]\!] = 0$. The spectral parameter $t$ appears in an exponential multiplied with $L$. To keep $tL$ dimensionless, we take $[\![t]\!] = -[\![L]\!] = -2$. The invariant of the lowest possible dimension is $\int d^n x \sqrt{g}\, \mathrm{tr}(f)$, where tr is a trace over the bundle indexes. This invariant has dimension $-n$, and has to be multiplied with $t^{-n/2}$ in the heat kernel expansion. This explains the lowest power of $t$ in expansion (4.9). In general, the mass dimension of the integrand appearing in $a_p(f, L)$ must be $p$. All possible invariant polynomials of the fields have even dimension. Consequently,

$$a_{2k+1}(f, L) = 0 \tag{4.29}$$

on manifolds without boundary. As we shall see below, on manifolds with boundaries non-trivial coefficients $a_{2k+1}(f, L)$ are allowed.

Now we are ready to write down several leading heat kernel coefficients.

$$a_0(f, D) = (4\pi)^{-n/2} \int_{\mathscr{M}} d^n x \sqrt{g}\, \mathrm{tr}(\alpha_0 f), \tag{4.30}$$

$$a_2(f, D) = (4\pi)^{-n/2} \frac{1}{6} \int_{\mathscr{M}} d^n x \sqrt{g}\, \mathrm{tr}\big(f(\alpha_1 E + \alpha_2 R)\big), \tag{4.31}$$

$$\begin{aligned} a_4(f, D) = (4\pi)^{-n/2} \frac{1}{360} \int_{\mathscr{M}} d^n x \sqrt{g}\, \mathrm{tr}\big(f\big(&\alpha_3 E_{;\mu}{}^{\mu} + \alpha_4 ER + \alpha_5 E^2 + \alpha_6 R_{;\mu}{}^{\mu} \\ &+ \alpha_7 R^2 + \alpha_8 R_{\mu\nu} R^{\mu\nu} + \alpha_9 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \alpha_{10} \Omega_{\mu\nu} \Omega^{\mu\nu}\big)\big). \end{aligned} \tag{4.32}$$

Here $\alpha_0$–$\alpha_{10}$ are undetermined dimensionless constants. This is about all the invariance properties can give. To determine the values of the constants we shall use other methods.

**Product of Base Manifolds and Universality of $\boldsymbol{\alpha_k}$** First let us consider the case when $\mathscr{M}$ is a direct product of two manifolds, $\mathscr{M} = \mathscr{M}_1 \times \mathscr{M}_2$, with coordinates $x_1$ and $x_2$ respectively. Let also $L$ be a sum of two operators $L_1$ and $L_2$ acting independently on $\mathscr{M}_1$ and $\mathscr{M}_2$, $L = L_1 \otimes 1 + 1 \otimes L_2$. This implies that the bundle indices are also independent. One can think of a vector Laplacian as an example.

One can write symbolically $\exp(-tL) = \exp(-tL_1) \otimes \exp(-tL_2)$. Next we multiply both sides of this equation by $f(x) = f_1(x_1)f_2(x_2)$, take the trace, and perform the asymptotic expansion in $t$ to get

$$a_k(x; L) = \sum_{p+q=k} a_p(x_1; L_1)a_q(x_2; L_2). \tag{4.33}$$

Let us take $\mathscr{M}_1$ being a unit one-dimensional circle $S^1$ and $L_1 = -\partial_1^2$. We keep $\mathscr{M}_2$ and $L_2$ arbitrary. According to (4.14) the only non-vanishing coefficient $a_p(x_1; -\partial_1^2)$ is $a_0(x_1; -\partial_1^2) = (4\pi)^{-1/2}$. Equation (4.33) yields

$$a_k(x; 1 \otimes L_2 - \partial_1^2 \otimes 1) = (4\pi)^{-1/2} a_k(x_2; L_2) \tag{4.34}$$

for all $k$ and $L_2$. All geometric invariants associated with $1 \otimes L_2 - \partial_1^2 \otimes 1$ are the same as for $L_2$ but taken in a lower dimension. Since we have extracted a power of $(4\pi)^{-n/2}$ in the formulae (4.30)–(4.32) explicitly, we conclude that *the constants $\alpha_j$ do not depend on the dimension of the manifold*. Therefore, the constants $\alpha_j$ are *universal* for any manifold, any vector bundle, and any Laplacian $L$. This is an extremely important property which allows us to make conclusions about the heat trace asymptotics in any dimensions by considering just simplest low-dimensional examples.

**Simplest Base Manifolds** Let us put $f \equiv 1$ and take the operator on a torus $T^n$. Equation (4.14) immediately gives

$$\alpha_0 = 1. \tag{4.35}$$

Next we consider scalar Laplacians on unit spheres $S^2$ and $S^3$. In the both cases $E = 0$ and $\Omega_{\mu\nu} = 0$, so that only the Riemann curvature given by (1.84) with $C = 1$ contributes. Since

$$\operatorname{vol} S^2 = 4\pi, \qquad \operatorname{vol} S^3 = 2\pi^2 \tag{4.36}$$

(see Exercise 4.3) the leading terms in expansions (4.15), (4.16) confirm the value (4.35) for $\alpha_0$. On unit $S^n$ one has: $R = n(n-1)$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 2n(n-1)$, $R_{\mu\nu}R^{\mu\nu} = n(n-1)^2$. By using these values we see that both expressions, (4.15) and (4.16), give the same value

$$\alpha_2 = 1 \tag{4.37}$$

for an undetermined constant in (4.31). Next we compare the term proportional to $t$ in expansion (4.15) to general expression (4.32) for $a_4$. We obtain

$$2\alpha_7 + \alpha_8 + 2\alpha_9 = 12. \tag{4.38}$$

The same procedure applied to the $t^{1/2}$-term of the heat kernel expansion (4.16) on $S^3$ yields

$$3\alpha_7 + \alpha_8 + \alpha_9 = 15. \tag{4.39}$$

Let us return to the product formula (4.33). Now we impose no restrictions on $\mathscr{M}_1$ and $\mathscr{M}_2$ and take $L_1$ and $L_2$ being scalar Laplacians. Let $R_1$ and $R_2$

be scalar curvatures on $\mathscr{M}_1$ and $\mathscr{M}_2$, respectively. Then the scalar curvature of $\mathscr{M} = \mathscr{M}_1 \times \mathscr{M}_2$ is $R = R_1 + R_2$. Let us take $k = 4$ in (4.33) and collect the terms on both sides of this equation which contain $R_1 \cdot R_2$. This gives

$$\frac{1}{360} 2\alpha_7 = \left(\frac{\alpha_2}{6}\right)^2 .$$

Constant $\alpha_2$ is calculated in (4.37). Therefore,

$$\alpha_7 = 5. \tag{4.40}$$

The conditions (4.38) and (4.39) yield

$$\alpha_8 = -2, \quad \alpha_9 = 2. \tag{4.41}$$

The terms in the heat kernel expansion which contain undifferentiated $E$ are relatively easy. Consider the case when $E$ is constant and proportional to the unit matrix. Then $K(f, L_0 - E; t) = e^{tE} K(f, L_0; t)$. By expanding this equation in $t$ and comparing to (4.30)–(4.32) we obtain

$$\frac{\alpha_1}{6} = \alpha_0, \quad \frac{\alpha_4}{360} = \frac{\alpha_2}{6}, \quad \frac{\alpha_5}{360} = \frac{\alpha_0}{2},$$

or

$$\alpha_1 = 6, \quad \alpha_4 = 60, \quad \alpha_5 = 180. \tag{4.42}$$

Later we shall also use an infinitesimal equation which governs the $E$-dependence of the heat kernel expansion

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} a_k(1, L - \epsilon f) = a_{k-2}(f, L). \tag{4.43}$$

**Dependence on Gauge Fields and Potentials** The part of the heat kernel expansion which does not depend on the curvature can be evaluated by a rather direct method. Let us consider a Laplacian on a unit torus $T^n$ acting on complex one-component scalar fields (i.e. on sections of a complex line bundle over $T^n$). The fibres are one-dimensional, so that we shall drop the symbol "tr" of the bundle trace until the end of this calculation. To calculate the heat trace of the operator $fe^{-tL}$ we use the normalized plane wave basis $(2\pi)^{-n/2} e^{ikx}$ where $k \in \mathbb{Z}^n$. In this basis

$$\begin{aligned} K(f, L; t) &= \sum_{k\in\mathbb{Z}^n} \int_{T^n} \frac{d^n x}{(2\pi)^n} e^{-ikx} f(x) \exp(-tL) e^{ikx} \\ &= \sum_{k\in\mathbb{Z}^n} \int_{T^n} \frac{d^n x}{(2\pi)^n} f(x) \exp(t((\nabla_\mu + ik_\mu)^2 + E)). \end{aligned} \tag{4.44}$$

On the second line of the equation above the operators act on a constant unit function (not written explicitly). Since to calculate the functional trace we have used a particular basis, the expression does not look gauge invariant. Explicit gauge invariance will be restored at the end of the calculation. We already know that the heat kernel expansion is organized according to the canonical mass dimension of the fields. Therefore, one can isolate in (4.44) the factor $e^{-tk^2}$ and expand the rest of the exponential in a power series of dimensional quantities $E$ and $\nabla$,

$$\begin{aligned}K(f,L;t) = \sum_{k\in\mathbb{Z}^n}\int_{T^n}\frac{d^nx}{(2\pi)^n}e^{-tk^2}f(x)\bigg(1 + t(\nabla^2 + E) - \frac{t^2}{2}4(k\nabla)^2\\ + \frac{t^2}{2}(\nabla^2\nabla^2 + \nabla^2 E + E\nabla^2 + E^2)\\ - \frac{4t^3}{6}\big((k\nabla)^2 E + E(k\nabla)^2 + (k\nabla)E(k\nabla)\big)\\ - \frac{4t^3}{6}\big((k\nabla)^2\nabla^2 + \nabla^2(k\nabla)^2 + (k\nabla)\nabla^2(k\nabla)\big)\\ + \frac{16t^4}{24}(k\nabla)^4 + \cdots\bigg).\end{aligned} \tag{4.45}$$

The above method is based on the rule of counting the dimensions, see the beginning of this subsection. Since the expansion in $t$ corresponds to an expansion in the canonical mass dimension, it is guaranteed, the existence of the expansion in $\nabla$ and $E$. The summation over $k$ can be performed by using the following asymptotic formulas:

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^n} e^{-tk^2} = \left(\frac{\pi}{t}\right)^{n/2} + \mathscr{O}(e^{-1/t}),\\ &\sum_{k\in\mathbb{Z}^n} e^{-tk^2}k_\mu k_\nu = \frac{1}{2t}\delta_{\mu\nu}\left(\frac{\pi}{t}\right)^{n/2} + \mathscr{O}(e^{-1/t}),\\ &\sum_{k\in\mathbb{Z}^n} e^{-tk^2}k_\mu k_\nu k_\rho k_\sigma = \frac{1}{4t^2}(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})\left(\frac{\pi}{t}\right)^{n/2} + \mathscr{O}(e^{-1/t}).\end{aligned} \tag{4.46}$$

This yields

$$\begin{aligned}K(f,L;t) = (4\pi t)^{-n/2}\int_{T^n} d^nx\, f(x)\bigg(1 + tE + \frac{t^2}{2}(\nabla^2\nabla^2 + \nabla^2 E + E\nabla^2 + E^2)\\ - \frac{t^2}{3}(\nabla^2 E + E\nabla^2 + \nabla^\mu E\nabla_\mu) - \frac{t^2}{3}(2\nabla^2\nabla^2 + \nabla^\mu\nabla^2\nabla_\mu)\\ + \frac{t^2}{6}(\nabla^\mu\nabla^\nu\nabla_\mu\nabla_\nu + \nabla^2\nabla^2 + \nabla^\mu\nabla^2\nabla_\mu) + \mathscr{O}(t^3)\bigg).\end{aligned} \tag{4.47}$$

All the derivatives combine into commutators and we finally get

$$\begin{aligned}K(f,L;t) = (4\pi t)^{-n/2}\int_{T^n} d^nx\, f(x)\bigg[1 + tE\\ + t^2\left(\frac{1}{2}E^2 + \frac{1}{6}E_{;\mu}^{\;\;\mu} + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu}\right)\bigg].\end{aligned} \tag{4.48}$$

This result confirms the values of $\alpha_0$ (4.35), of $\alpha_1$ and $\alpha_5$ (4.42), and also gives new values

$$\alpha_3 = 60, \quad \alpha_{10} = 30. \tag{4.49}$$

**The Method of Conformal Variations** One universal constant, namely $\alpha_6$, still remains to be calculated. This gives us an opportunity to introduce another powerful method. Namely, we shall use transformations of background fields under which *any* Laplacian $L$ transforms homogeneously, $L \to L_\sigma = e^{-2\sigma} L$, where $\sigma$ is a smooth function on $\mathscr{M}$. The transformation law for the metric is just the standard local Weyl transformation,

$$g^{\mu\nu} \to e^{-2\sigma} g^{\mu\nu}. \tag{4.50}$$

Transformations (4.50) are called conformal transformations and are related to interesting features of quantum theory on curved manifolds, see Sect. 8.5. To the linear order in $\sigma$ we have: $\delta g^{\mu\nu} = -2\sigma g^{\mu\nu}$ and $\delta g_{\mu\nu} = +2\sigma g_{\mu\nu}$. The rules

$$\begin{aligned}
&\delta\Gamma^\rho_{\mu\nu} = \delta^\rho_\mu \sigma_{;\nu} + \delta^\rho_\nu \sigma_{;\mu} - g_{\mu\nu}\sigma^{;\rho},\\
&\delta R_{\lambda\mu\nu\kappa} = 2R_{\lambda\mu\nu\kappa}\sigma + g_{\lambda\kappa}\sigma_{;\nu\mu} - g_{\mu\kappa}\sigma_{;\nu\lambda} - g_{\nu\lambda}\sigma_{;\kappa\mu} + g_{\mu\nu}\sigma_{;\kappa\lambda},\\
&\delta R = -2R\sigma - 2(n-1)\sigma_{;\mu}{}^{\mu}
\end{aligned} \tag{4.51}$$

follow from (4.50) and explicit definitions (1.4), (1.10) and (1.16). We remind, that the semicolon denotes covariant derivative, see (1.14). Since we requested that $L$ as a whole transforms homogeneously, the matrix valued functions $a^\mu$ and $b$ (see (3.1)) must also transform homogeneously, $\delta a^\mu = -2a^\mu\sigma$, $\delta b = -2b\sigma$. By using (3.3), we find

$$\begin{aligned}
&\delta\omega_\mu = \frac{1}{2}(2-n)\sigma_{;\mu},\\
&\delta\Omega_{\mu\nu} = 0,\\
&\delta E = -2E\sigma + \frac{1}{2}(n-2)\sigma_{;\mu}{}^{\mu}.
\end{aligned} \tag{4.52}$$

It is important to keep in mind that transformations (4.52) are *not* the Weyl transformations, in general. Note, that generic operators of the Laplace type need not transform homogeneously under the Weyl transformations.

Let us now study how the heat kernel transforms under the conformal variations that we have defined. To have an explicit small parameter we write $\sigma(x) = \epsilon f(x)$. Then

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathrm{Tr}\big(e^{-tL_{\epsilon f}}\big) = \mathrm{Tr}\big(2ftLe^{-tL}\big) = -2t\frac{d}{dt}K(f,L;t). \tag{4.53}$$

By expanding (4.53) in asymptotic power series in $t$ we note that

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} a_k(1, L_{\epsilon f}) = (n-k)a_k(f,L). \tag{4.54}$$

This is a powerful relation which we shall use in Sect. 5.7. With its help one can prove the following conformal relation (see Exercise 4.4):

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} a_{n-2}(e^{-2\epsilon f}F, L_{\epsilon f}) = 0, \tag{4.55}$$

where $F$ is a matrix-valued function. We apply (4.55) to $n=6$ and collect all terms containing $Ff_{;\mu\ \nu}^{\ \mu\ \nu}$. Obviously, such terms can only come from $E_{;\mu}^{\ \ \mu}$ and $R_{;\mu}^{\ \ \mu}$ in (4.32). The coefficients can be easily found from (4.51) and (4.52) and one concludes that $\alpha_6 = 12$.

This was the last universal constant which had to be determined. Now one can summarize our results for the first heat coefficients $a_0$, $a_2$ and $a_4$ on a compact manifold without boundaries

$$a_0(f,L) = (4\pi)^{-n/2}\int_{\mathscr{M}} d^nx\,\sqrt{g}\,\mathrm{tr}(f), \tag{4.56}$$

$$a_2(f,L) = (4\pi)^{-n/2}\frac{1}{6}\int_{\mathscr{M}} d^nx\,\sqrt{g}\,\mathrm{tr}\big(f(6E+R)\big), \tag{4.57}$$

$$\begin{aligned} a_4(f,L) = (4\pi)^{-n/2}\frac{1}{360}\int_{\mathscr{M}} d^nx\,\sqrt{g}\,\mathrm{tr}\big(f\big(60E_{;\mu}^{\ \ \mu} + 60ER + 180E^2 \\ + 12R_{;\mu}^{\ \ \mu} + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 30\Omega_{\mu\nu}\Omega^{\mu\nu}\big)\big). \end{aligned} \tag{4.58}$$

As we shall see in Sect. 7.5 these results are of crucial importance for quantum field theories in an external background field in 4 space-time dimensions.

**Examples for Various Spins** First, let us consider the scalar Laplacian $\Delta^{(0)} = -D_\mu D^\mu + \xi R$ acting on charged scalar fields $D_\mu = \nabla_\mu + ieA_\mu$, with $A_\mu$ being an Abelian gauge field with the field strength $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. This operator takes the canonical form (3.2) with $\omega_\mu = ieA_\mu$, $\Omega_{\mu\nu} = ieF_{\mu\nu}$ and $E = -\xi R$. Let us put the smearing function $f=1$. Then

$$a_2(\Delta^{(0)}) = (4\pi)^{-n/2}\frac{1}{6}\int_{\mathscr{M}} d^nx\,\sqrt{g}(1-6\xi)R, \tag{4.59}$$

$$\begin{aligned} a_4(\Delta^{(0)}) = (4\pi)^{-n/2}\frac{1}{360}\int_{\mathscr{M}} d^nx\,\sqrt{g}\big((180\xi^2 - 60\xi + 5)R^2 \\ - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 30e^2F_{\mu\nu}F^{\mu\nu}\big). \end{aligned} \tag{4.60}$$

The next example is the vector Laplacian $\Delta^{(1)}$ defined in (3.5). In this case the bundle indices are the vector indices, $E_\mu^{\ \nu} = -R_\mu^{\ \nu}$, and the field strength $\Omega_{\mu\nu}$ is defined through the commutator of Riemannian covariant derivatives

$$[\nabla_\mu, \nabla_\nu]v_\rho = (\Omega_{\mu\nu})_\rho^{\ \sigma} v_\sigma$$

yielding

$$(\Omega_{\mu\nu})_\rho^{\ \sigma} = -R^\sigma_{\ \rho\mu\nu}.$$

Now we are ready to calculate the traces

$$\mathrm{tr}(E) = -R, \qquad \mathrm{tr}\, ER = R^2,$$

$$\mathrm{tr}\, E^2 = R_{\mu\nu}R^{\mu\nu}, \qquad \mathrm{tr}\, \Omega_{\mu\nu}\Omega^{\mu\nu} = -R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}.$$

All terms in (4.57) and (4.58) which depend on the Riemann curvature only are unit matrices in the bundle indices, so that the corresponding traces give a multiplier of $n$. Finally, we have

$$a_2(\Delta^{(1)}) = (4\pi)^{-n/2}\frac{n-6}{6}\int_{\mathcal{M}} d^n x\, \sqrt{g}\, R, \tag{4.61}$$

$$a_4(\Delta^{(1)}) = (4\pi)^{-n/2}\frac{1}{360}\int_{\mathcal{M}} d^n x\, \sqrt{g}\big((5n-60)R^2 + (180-2n)R_{\mu\nu}R^{\mu\nu} + (2n-30)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\big). \tag{4.62}$$

The last example is the spin 1/2 Laplacian $\Delta^{(1/2)} = \not{D}^2(B)$ where $\not{D}(B) = i\gamma^\mu(\nabla^{[s]}_\mu + gB_\mu)$ with a non-Abelian gauge field $B_\mu$ taken in an $N$-dimensional unitary representation of the Lie algebra corresponding to the gauge group (one can take, e.g., the fundamental representation of $su(N)$). The corresponding field strength $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + g[B_\mu, B_\nu]$ is also a matrix in the same representation. In this case the bundle indices are both gauge and spinor indices. The covariant derivative $\nabla_\mu$ is exactly $D_\mu(B)$. After some algebra one gets

$$E = \frac{g}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu} - \frac{1}{4}R,$$
$$\Omega_{\mu\nu} = \frac{1}{4}\gamma^a\gamma^b R_{ab\mu\nu} + gF_{\mu\nu}.$$

Let us calculate traces of the invariants which appear in (4.57) and (4.58). Let $r = 2^{\lfloor n/2\rfloor}$ be the number of components of a Dirac spinor in $n$ dimensions, and let $\widetilde{\mathrm{tr}}$ be a trace over the gauge indices. Then

$$\mathrm{tr}\, R = NrR, \qquad \mathrm{tr}\, 6E = -(3/2)NrR,$$
$$\mathrm{tr}\, 60ER = -15NrR^2,$$
$$\mathrm{tr}\, 180E^2 = \frac{45}{4}NrR^2 - 90rg^2\widetilde{\mathrm{tr}}(F_{\mu\nu}F^{\mu\nu}),$$
$$\mathrm{tr}\, 30\Omega_{\mu\nu}\Omega^{\mu\nu} = -\frac{15}{4}NrR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 30rg^2\widetilde{\mathrm{tr}}(F_{\mu\nu}F^{\mu\nu}).$$

By collecting everything together, one obtains

$$a_2(\Delta^{(1/2)}) = -(4\pi)^{-n/2}\frac{Nr}{12}\int_{\mathcal{M}} d^n x\, \sqrt{g}R, \tag{4.63}$$

$$a_4(\Delta^{(1/2)}) = (4\pi)^{-n/2}\frac{r}{360}\int_{\mathcal{M}} d^n x\, \sqrt{g}\left(N\left(\frac{5}{4}R^2 - 2R_{\mu\nu}R^{\mu\nu} - \frac{7}{4}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) - 60g^2\widetilde{\mathrm{tr}}(F_{\mu\nu}F^{\mu\nu})\right). \tag{4.64}$$

## 4.5 Base Manifolds with Boundaries

**Simple Examples** Let us take the standard scalar Laplacian on the interval $\mathscr{I} = [0, l]$ with Dirichlet or Neumann boundary conditions at the endpoints. The spectrum is given by (3.27) and (3.28). The corresponding heat traces can be easily evaluated by reducing them to (4.13). We obtain

$$K(\Delta_{\mathscr{I},D}t) = \sum_{k>0} \exp\left(-t\frac{k^2\pi^2}{l^2}\right) = \frac{1}{2}\left(\frac{l}{\sqrt{\pi t}} - 1\right) + \mathscr{O}(e^{-1/t}), \qquad (4.65)$$

$$K(\Delta_{\mathscr{I},N}t) = \sum_{k\geq 0} \exp\left(-t\frac{k^2\pi^2}{l^2}\right) = \frac{1}{2}\left(\frac{l}{\sqrt{\pi t}} + 1\right) + \mathscr{O}(e^{-1/t}). \qquad (4.66)$$

We see, that these asymptotic expansions have the form (4.9), but in contrast to the case without boundaries also a coefficient $a_1$ with an odd index appears.

On some manifolds one can derive an explicit expression for the heat kernel by the method of images. Consider a half-space $\mathscr{M} = \mathbb{R}^{n-1} \times \mathbb{R}_+$. Then the kernel

$$K_{D,N}(x, y|t) = (4\pi t)^{-n/2}\left[\exp\left(-\frac{(x-y)^2}{4t}\right) \mp \exp\left(-\frac{(x-y^*)^2}{4t}\right)\right], \qquad (4.67)$$

where $y^* = (y^1, \dots, y^{n-1}, -y^n)$, satisfies the heat equation for both $x$, $y$ inside $\mathscr{M}$ and Dirichlet (respectively, Neumann) boundary conditions if $x$ or $y$ is on the boundary. For curved boundaries, if both $x$ and $y$ are near the boundary the heat kernel may be approximated by a combination of two terms, one depending on the length of the geodesic going directly from $x$ to $y$, and another one—of the geodesic reflected at the boundary. Therefore, one can in principle write down an ansatz similar to (4.18) and apply the DeWitt recursion procedure [185]. However, for practical use this method appears to be too complicated.

**Structure of the Heat Kernel Coefficients** The method of Gilkey works perfectly well also on manifolds with boundaries. The only problem is that the number of independent invariants increases, and expressions for the heat kernel coefficients grow longer. Here we illustrate the method with the examples of Dirichlet

$$\varphi|_{\partial\mathscr{M}} = 0 \qquad (4.68)$$

and Robin (or modified Neumann)

$$(\nabla_n + \mathscr{S})\varphi|_{\partial\mathscr{M}} = 0 \qquad (4.69)$$

boundary conditions (see (3.43) and (3.44) where the notations are explained). First of all, we have to discuss the general structure of heat kernel coefficients. These coefficients are local. This means they contain bulk and boundary contributions. The bulk parts "do not see the boundary", i.e., if the smearing function vanishes in a vicinity of the boundary, the heat kernel coefficients look precisely as in the previous section. The boundary contributions are constructed from bulk and specific boundary invariants which were described in Sect. 1.8. Roughly speaking, in addition to former invariants one can use normal derivatives and the extrinsic curvature $K_{ij}$. The arguments based on canonical mass dimensions of the fields still work, but one has to define dimensions of the new invariants. For example, for the extrinsic curvature we have $[\![K_{ij}]\!] = 1$ because $K$ contains one derivative of the metric. For Robin boundary conditions $[\![\mathscr{S}]\!] = 1$ because $\mathscr{S}$ appears in the condition (4.69) in a linear combination with a derivative. Since we have new invariants of odd dimension, the coefficients $a_{2k+1}$ need not vanish. Therefore, we can write for Dirichlet boundary conditions

$$a_0(f, L) = (4\pi)^{-n/2} \int_{\mathscr{M}} d^n x \sqrt{g} \operatorname{tr}(f), \tag{4.70}$$

$$a_1(f, L) = (4\pi)^{-(n-1)/2} \int_{\partial\mathscr{M}} d^{n-1} x \sqrt{h} \operatorname{tr} \beta_1^D(f), \tag{4.71}$$

$$a_2(f, L) = (4\pi)^{-n/2} \frac{1}{6} \bigg[ \int_{\mathscr{M}} d^n x \sqrt{g} \operatorname{tr}\big(f(6E + R)\big) + \int_{\partial\mathscr{M}} d^{n-1} x \sqrt{h} \operatorname{tr}\big(\beta_2^D f K_j^j + \beta_3^D f_{;n}\big) \bigg] \tag{4.72}$$

and for Robin (Neumann) boundary conditions

$$a_0(f, L) = (4\pi)^{-n/2} \int_{\mathscr{M}} d^n x \sqrt{g} \operatorname{tr}(f), \tag{4.73}$$

$$a_1(f, L) = (4\pi)^{-(n-1)/2} \int_{\partial\mathscr{M}} d^{n-1} x \sqrt{h} \operatorname{tr} \beta_1^N(f), \tag{4.74}$$

$$a_2(f, L) = (4\pi)^{-n/2} \frac{1}{6} \bigg[ \int_{\mathscr{M}} d^n x \sqrt{g} \operatorname{tr}\big(f(6E + R)\big) + \int_{\partial\mathscr{M}} d^{n-1} x \sqrt{h} \operatorname{tr}\big(\beta_2^N f K_j^j + \beta_3^N f_{;n} + \beta_4^N f \mathscr{S}\big) \bigg]. \tag{4.75}$$

As before, $\beta_j^{D,N}$ are undetermined constants.

**Products of Base Manifolds** We may use the product formula (4.33) on manifolds with boundary (though it now contains $\delta$-function like contributions on both sides). Let us choose $\mathscr{M}_1$ being $S^1$. The boundary has the product structure as well, $\partial\mathscr{M} = S^1 \times \partial\mathscr{M}_2$. One can also choose the boundary conditions consistent with the product structure: it is enough to require that $\mathscr{S}$ in (4.69) does not depend on the coordinate $x^1$ of $S^1$. Then formula (4.34) is true, and it tells us that the constants $\beta_j^{D,N}$ do not depend on $n$. (This property, however, does not hold for boundary conditions which are not consistent with the product structure. The example is related to non-local Atiyah-Patodi-Singer boundary conditions [14–16]. The heat kernel coefficients in this case depend on $n$ in a rather complicated way.)

To define $\beta_1^{D,N}$, let us consider a Laplacian on the trivial base manifold $\mathscr{M} = \mathscr{I} = [0, l]$. The boundary consists of two points, and Eqs. (4.71) and (4.74) yield

$$a_1(\Delta_{\mathscr{I},D}) = 2\beta_1^D, \quad a_1(\Delta_{\mathscr{I},N}) = 2\beta_1^N.$$

We compare these expression with (4.65) and (4.66) to obtain

$$\beta_1^N = -\beta_1^D = \frac{1}{4}. \tag{4.76}$$

**Isospectral Operators** Let us take another one-dimensional example which provides the value of $\beta_4^N$. This example appears to be more complicated but it is rather instructive. As above, the base manifold is the interval $[0, l]$. Let us introduce first order operators

$$D_\pm = \partial_x \pm V(x), \tag{4.77}$$

where $V(x)$ is a smooth function of the coordinate, and define with their help two Laplacians

$$L_1 = -D_+ D_-, \qquad L_2 = -D_- D_+. \tag{4.78}$$

As a consequence, the operators satisfy the so-called 'intertwining' relations, $D_- L_1 = L_2 D_-$, $D_+ L_2 = L_1 D_+$. Let $\varphi_\lambda^{(1)}$ be an eigenfunction of $L_1$, $L_1 \varphi_\lambda^{(1)} = \lambda \varphi_\lambda^{(1)}$ for $\lambda \neq 0$. Then

$$\varphi_\lambda^{(2)} = D_- \varphi_\lambda^{(1)} \tag{4.79}$$

is an eigenfunction of $L_2$ corresponding to the same eigenvalue $\lambda$. Indeed, $L_2 \varphi_\lambda^{(2)} = -D_- D_+ D_- \varphi_\lambda^{(2)} = D_- \lambda \varphi_\lambda^{(1)} = \lambda \varphi_\lambda^{(2)}$. Acting with $D_+$ on $\varphi_\lambda^{(2)}$ gives $\varphi_\lambda^{(1)}$ up to a normalization,

$$\varphi_\lambda^{(1)} \propto D_+ \varphi_\lambda^{(2)}. \tag{4.80}$$

We conclude, that the spectra of $L_1$ and $L_2$ can be identified up to zero modes of $D_\pm$ (i.e. $L_1$ and $L_2$, are isospectral up to zero modes).

We use the boundary conditions which are consistent with the intertwining relations and suppose that $\varphi_\lambda^{(2)}$ satisfies Dirichlet boundary conditions,

$$\varphi_\lambda^{(2)}|_{\partial\mathscr{M}} = 0 \tag{4.81}$$

for all $\lambda$. Then Eq. (4.79) yields Robin boundary conditions for $\varphi_\lambda^{(1)}$,

$$(\partial_x - V)\varphi_\lambda^{(1)}|_{\partial\mathscr{M}} = 0. \tag{4.82}$$

The boundary conditions (4.81) and (4.82) allow us to integrate by parts without introducing boundary terms in inner product (3.13). If $\varphi$ and $\varphi'$ satisfy (4.82), one can easily check that $(\varphi', L_1\varphi) = (D_-\varphi', D_-\varphi)$. A similar relation holds for $L_2$. Consequently, $L_1$ and $L_2$ are symmetric non-negative operators whose zero modes coincide with zero modes of $D_-$ and $D_+$, respectively.

The zero modes of $D_\pm$ are:

$$\varphi_0^\pm = c_\pm \exp\left(\mp \int_0^x V(y)\, dy\right), \tag{4.83}$$

where $c$ is a constant. The Robin condition (4.82) does not restrict $c_-$, while the Dirichlet condition yields $c_+ = 0$. Therefore, there are no zero modes in the Dirichlet sector (for $L_2$), but there is always one normalized zero mode in the Robin sector (for $L_1$). We arrive at a remarkable relation between the heat kernels

$$K(L_1; t) - 1 = K(L_2; t), \tag{4.84}$$

where $-1$ on the left hand side is a contribution from the zero mode of $L_2$. Besides this zero mode, the spectra coincide, and so do the heat kernels. By expanding (4.84) in an asymptotic series in $t$, we obtain

$$a_1(L_1) - 1 = a_1(L_2), \tag{4.85}$$

$$a_k(L_1) = a_k(L_2) \quad \text{for } k \neq 1. \tag{4.86}$$

Let us write our operators more explicitly, $L_{1,2} = -(\partial_x^2 \mp (\partial_x V) - V^2)$. All associated geometric invariants are trivial, $\omega_\mu = 0$, and

$$E_{1,2} = \mp(\partial_x V) - V^2. \tag{4.87}$$

For the operator $L_1$ we should also define $\mathscr{S}$. It reads

$$\mathscr{S}(x = 0) = -V(0), \qquad \mathscr{S}(x = l) = V(l). \tag{4.88}$$

Let us remind, that $\partial_n$ is a derivative with respect to an *inward* pointing unit vector, to that $\partial_n|_{x=0} = \partial_x$, $\partial_n|_{x=l} = -\partial_x$. Now we are ready to study the consequences of (4.85) and (4.86). For $k = 0$ Eq. (4.86) is satisfied trivially. The condition (4.85) confirms the values (4.76) of $\beta_1^{N,D}$. For $k = 2$ Eq. (4.86) yields

$$0 = (4\pi)^{-1/2} \frac{1}{6} (V(l) - V(0))(12 - \beta_4^N)$$

or

$$\beta_4^N = 12. \tag{4.89}$$

To get this equality one had to integrate in the bulk parts in (4.72) and (4.75) the term $\partial_x V$ in $E_1$ and $E_2$.

**Using Conformal Maps** All other constants will be defined by using conformal variations. The relations (4.51) and (4.52) derived in the previous section remain true. One has to define conformal properties of specific “boundary” variables. Conformal transformations preserve angles. Therefore, the normal vector to the boundary will remain normal, but its length will change. To compensate this change under infinitesimal conformal transformations of the metric (4.50) we require $\delta n_\mu = \sigma n_\mu$ and $\delta n^\mu = -\sigma n^\mu$. The extrinsic curvature transforms as

$$\delta K_j^j = -\sigma K_j^j - (n - 1)\sigma_{;n} \tag{4.90}$$

(see Exercise 4.6).

The Dirichlet boundary condition (4.68) is obviously conformally invariant. Consider the Robin boundary conditions (4.69). From Eq. (4.52) we find

$$\delta(\nabla_n) = \delta(n^\mu \nabla_\mu) = -\sigma \nabla_n + \frac{1}{2}(2-n)\sigma_{;n}.$$

To achieve conformal invariance of the boundary value problem we have to define conformal transformations of $\mathscr{S}$ in such way that the inhomogeneous term in the equation above is canceled,

$$\delta\mathscr{S} = -\sigma\mathscr{S} - \frac{1}{2}(2-n)\sigma_{;n}. \tag{4.91}$$

Then $\delta(\nabla_n + \mathscr{S}) = -\sigma(\nabla_n + \mathscr{S})$, and the functions which satisfied the Robin condition before the conformal transformation, will also satisfy the conformally transformed Robin condition.

Consider the scalar Laplacian $L = \Delta$ on a two-dimensional disc of a unit radius ($n = 2$). Put $f = 1$. The extrinsic curvature of the boundary is given by (1.92),

$$K_j^j = 1. \tag{4.92}$$

For Dirichlet boundary conditions Eq. (4.72) gives

$$a_2(\Delta_{\text{disc},D}) = \frac{1}{12}\beta_2^D, \tag{4.93}$$

where the only contribution comes from the extrinsic curvature. For Robin boundary conditions with $\mathscr{S} = 0$ we have similarly from (4.72)

$$a_2(\Delta_{\text{disc},N}) = \frac{1}{12}\beta_2^N. \tag{4.94}$$

We can also consider the same operator on a two-dimensional hemisphere and use the fact that the hemisphere is conformally equivalent to the disc. To proceed, the operators for these base manifolds will be denoted as $\Delta_{\text{disc},D(N)}$ and $\Delta_{\text{h.s.}D(N)}$. The conditions $E = 0$ and $\mathscr{S} = 0$ are conformally invariant (see (4.52) and (4.91), respectively). According to (4.54) the coefficient $a_2$ is conformally invariant in $n = 2$,

$$a_2(\Delta_{\text{disc},D(N)}) = a_2(\Delta_{\text{h.s.},D(N)}).$$

The extrinsic curvature of the boundary of the hemisphere vanishes (see Exercise 1.13). Therefore, the only contribution to the heat kernel coefficient $a_2$ comes from the scalar curvature $R = 2$,

$$a_2(\Delta_{\text{h.s.},D}) = a_2(\Delta_{\text{h.s.},N}) = \frac{1}{6}. \tag{4.95}$$

We conclude that

$$\beta_2^D = \beta_2^N = 2. \tag{4.96}$$

Let us now return to the generic case. By collecting all boundary terms with $f_{;n}$ in the conformal variations (4.54) in (4.72) and (4.75) (by taking into account total derivatives in the bulk) we obtain

$$(-3n+6)f_{;n} = (n-2)\beta_3^D f_{;n}, \tag{4.97}$$

$$(3n-6)f_{;n} = (n-2)\beta_3^N f_{;n}, \tag{4.98}$$

where we have used (4.96) and (4.89). These relations yield

$$\beta_3^D = -3, \quad \beta_3^N = 3. \tag{4.99}$$

This completes the calculations of three leading heat kernel coefficients for Dirichlet and generalized Neumann (Robin) boundary conditions.

There is another more economic way to obtain the results above, see Exercise 4.7.

## 4.6 Base Manifolds with Codimension One Defects

There are numerous physical applications when background spaces are smooth everywhere except some internal hypersurfaces $\Sigma$. Examples discussed in Sect. 1.9 are related to various kinds of defects of the geometry. Although curvature at the defects cannot be defined locally, integrals of curvature invariants still may be meaningful.

The heat kernel coefficients in the asymptotic expansion (4.9) are integrals of geometrical characteristics of the base manifolds. Therefore, the asymptotics may be well-defined in the presence of the defects. A natural question is the form of the heat coefficients in this case.

We begin with co-dimension one defects. A singular manifold of this type is constructed from two smooth manifolds $\mathscr{M}_+$ and $\mathscr{M}_-$ glued together along their common boundary $\Sigma$. The fields on $\mathscr{M}_+$ and $\mathscr{M}_-$ should also be glued together in some way. Let us define on $\Sigma$ a set of data, $\varphi^\pm$, $\varphi^\pm_{;n}$, which represent, respectively, the limiting values of a field $\varphi$ and its normal derivative when one approaches $\Sigma$ either from inside $\mathscr{M}_+$ or $\mathscr{M}_-$. On $\mathscr{M}_+$ and $\mathscr{M}_-$ the field is assumed to be smooth outside $\Sigma$. If we are interested in spectral problems for a Laplace type operator, the following suitable matching conditions between the data on $\Sigma$ have to be introduced

$$\begin{aligned} \varphi^+_{;n} &= S_{++}\varphi^+ + S_{+-}\varphi^-, \\ \varphi^-_{;n} &= S_{-+}\varphi^+ + S_{--}\varphi^-. \end{aligned} \tag{4.100}$$

Here $S_{\pm\mp}$ are maps between the restrictions of the fiber bundles $\mathscr{E}_\pm$ over $\mathscr{M}_\pm$ to $\Sigma$. Note that we do not imply any relations between $\mathscr{E}_\pm$. Even dimensions of $\mathscr{E}_\pm$ may not coincide (the case of completely different fields interacting across the brane $\Sigma$). To make sure that the number of conditions is correct, let us put $S_{+-} = S_{-+} = 0$. Then (4.100) is a pair of Robin boundary conditions on $\mathscr{M}_+$ and $\mathscr{M}_-$ which indeed specify a well-defined spectral problem on each of the manifolds.

There exist much more invariants associated with the spectral problems of this type than in the case of Dirichlet or Neumann boundary value problems. Since most of geometric quantities may jump on $\Sigma$, we have the limiting values of $E$ from two sides of $\Sigma$ instead of just one value of $E$ on the boundary, two extrinsic curvatures, etc. However, the calculations of the heat trace asymptotics are not much harder than the corresponding calculations for local boundary conditions. To illustrate this point, let us consider a particular case of a delta-function potential concentrated on the brane. Take the operator

$$L = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + E(x) + V(x)\delta_\Sigma), \tag{4.101}$$

where, for simplicity, we assume that $E(x)$ and the connection in $\nabla$ are smooth across $\Sigma$, but the normal derivative of the metric may jump. The $\delta$-function is defined such that for any smooth function $f$ on $\mathscr{M}$

$$\int_{\mathscr{M}} f(x)\delta_\Sigma(x)\sqrt{\det g_{\mu\nu}}\, d^n x = \int_\Sigma f(x)\sqrt{\det g_{ik}}\, d^{n-1}x. \tag{4.102}$$

The operator (4.101) does not define yet any well-posed spectral problem. Let us discuss what kind of matching conditions should be imposed. Because a product of $\delta_\Sigma$ with a function which has a discontinuity at $\Sigma$ is meaningless we have to request

$$\varphi^+ = \varphi^-. \tag{4.103}$$

To determine the second matching condition we use the Riemann normal coordinates (1.88) near $\Sigma$, choose a small part $\sigma \subset \Sigma$, and integrate the eigenvalue equation $L\varphi = \lambda\varphi$ over a cylinder $\sigma \times [-\epsilon, \epsilon]$. In the limit $\epsilon \to 0$ only the contributions from the first and the third terms in (4.101) survive, and we arrive at the condition

$$\int_\sigma \left((\varphi^+_{;n} + \varphi^-_{;n}) + V\varphi\right)\sqrt{\det g_{ij}}\, d^{n-1}x = 0. \tag{4.104}$$

For an arbitrary $\sigma$ this equation implies that

$$(\varphi^+_{;n} + \varphi^-_{;n}) + V\varphi = 0. \tag{4.105}$$

We come to the following formulation of the spectral problem for operator (4.101) on a brane: it is a standard spectral problem inside $\mathscr{M}_+$ and $\mathscr{M}_-$ for the smooth part of $L$ (without the delta-term) supplemented by two matching conditions (4.103) and (4.105).

Let us now consider the heat trace asymptotics. Obviously, due to the locality of the heat kernel coefficients, they may be expressed through the integrals over $\mathscr{M}_\pm$ and $\Sigma$ of the invariants constructed from usual bulk geometric quantities ($E$, Riemann tensor, field strength, etc.) and the new geometric quantities on the brane, namely, $K^\pm_{ij}$ and $V$. Each integrand has a definite mass dimension, as described in Sect. 4.5, and $[\![V]\!] = 1$. There are several obvious statements which reduce the number of relevant invariants considerably:

1. In the smooth limit, i.e., when $K^+_{ij} + K^-_{ij} = 0$ and $V = 0$ the brane contribution should vanish. This rules out the only possible invariant $\int_\Sigma d^{n-1}x\sqrt{h}\,\mathrm{tr}(f)$ which may appear in $a_1$. Neither $a_0$ nor $a_1$ have brane contributions.
2. The heat kernel coefficients must be invariant under exchange $\mathscr{M}_+$ and $\mathscr{M}_-$. This rules out the term $(K_i^{+i} - K_i^{-i})$ which could have appeared in $a_2$.

By assuming that the smearing function $f$ is smooth across $\Sigma$, we arrive at the following expression for the heat kernel coefficient $a_2$:

$$a_2(f,L) = (4\pi)^{-n/2}\frac{1}{6}\Big[\int_{\mathscr{M}_+\cup\mathscr{M}_-} d^n x\,\sqrt{g}\,\mathrm{tr}\big(f(6E+R)\big) + \int_\Sigma d^{n-1}x\,\sqrt{h}\,\mathrm{tr}\big(\beta_2^\Sigma f(K_j^{+j}+K_j^{-j}) + \beta_4^\Sigma fV\big)\Big]. \tag{4.106}$$

One must determine the constants $\beta_2^\Sigma$ and $\beta_4^\Sigma$. The numeration of the constants will become clear in a moment. Consider the case when $\mathscr{M}_+$ and $\mathscr{M}_-$ are two identical manifolds, and the restrictions of $L$ on these manifolds coincide. In other words, we have two mirror images of a manifold with boundary $\Sigma$. The operator $L$ commutes with the reflections $\mathscr{M}_+ \leftrightarrow \mathscr{M}_-$. Consequently, all eigenfunctions can be divided into two sets, symmetric and antisymmetric ones. It is easy to show that antisymmetric functions satisfy Dirichlet boundary conditions on $\Sigma$, while symmetric functions satisfy Robin boundary conditions (3.42), (3.44) with $\mathscr{S} = \frac{1}{2}V$. These properties follow from the matching conditions above, Eq. (4.105), see Exercise 4.8. If we define the heat kernel coefficients $a_k(f,L)_{D,R}$ for the restriction of $L$ on $\mathscr{M}$ with the boundary conditions that we have just defined, the following statement is obvious

$$a_k(f,L) = a_k(f,L)_N + a_k(f,L)_D. \tag{4.107}$$

Taking $k=2$ in this formula, and using (4.72) and (4.75), we obtain:

$$\beta_4^\Sigma = 6, \quad \beta_2^\Sigma = 2. \tag{4.108}$$

It is interesting to note, that simply substituting $E = \delta_\Sigma V$ in the bulk integral in (4.106) reproduces correctly the $\beta_4^\Sigma$ coefficient. This property holds for all linear in $V$ terms having a "smooth" origin, but is, of course, lost for more singular terms, such as $V^2$, for example.

To calculate higher heat kernel coefficients in the presence of codimension 1 defects one can use all other techniques introduced in Sects. 4.2 and 4.5.

## 4.7 Base Manifolds with Conical Singularities

**Heat Kernel on a Cone** Consider now the heat kernel asymptotics on manifolds with conical singularities which are co-dimension 2 defects. The starting point is the heat kernel of the scalar Laplace operator $L = -\partial_\mu^2$ on a two-dimensional cone $C_\beta$ with metric (1.96). The heat kernel on $C_\beta$ allows a simple representation in terms of the heat kernel on a two-plane $\mathbb{R}^2$,

$$K\big(x(\tau),x'(0)|t\big) = \frac{1}{4\pi t}e^{-(x(\tau)-x'(0))^2/4t}, \tag{4.109}$$

see (4.17). In polar coordinates $(\rho\cos\tau, \rho\sin\tau)$

$$(x(\tau)-x'(0))^2 \equiv \rho^2 + (\rho')^2 - 2\rho\rho'\cos\tau. \tag{4.110}$$

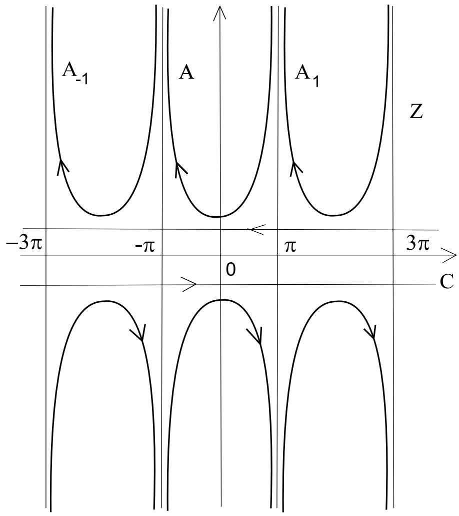


**Fig. 4.1** The integration contour in Eq. (4.113)

Without any loss of generality and on the base of the rotation symmetry we put the angular coordinate of one of the points equal to zero.

To find an appropriate representation for the kernel on $C_\beta$ we first consider (4.109) as a function of $\tau$ and use the Cauchy theorem to write

$$K\big(x(\tau), x'(0)|t\big) = \frac{1}{2\pi i}\oint \frac{1}{z-\tau} K\big(x(z), x'(0)|t\big)\, dz, \tag{4.111}$$

where the contour in the complex plane goes in the positive direction along a circle with the center at the point $z = \tau$. The integration contour can be transformed into two lines $C$ parallel to the real axis. The lines can be further replaced by a sequence of congruent contours $A = A_0 \bigcup_n A_n$, $n = \pm 1, \pm 2, \ldots$, see Fig. 4.1. Each $A_n$ consists of two parts: in the upper plane it goes from $(2n+1)\pi - \epsilon + i\infty$ to $(2n-1)\pi + \epsilon + i\infty$, in the lower plane from $(2n-1)\pi + \epsilon - i\infty$ to $(2n+1)\pi - \epsilon - i\infty$. The contours $A_n$ are chosen so that to have a periodic structure and to ensure convergence of the integrals. The latter property can be checked by using (4.109) and (4.110). Let us introduce a kernel

$$K_\infty\big(x(\tau), x'(0)|t\big) \equiv \frac{1}{2\pi i}\int_{A_0} \frac{1}{z-\tau} K\big(x(z), x'(0)|t\big)\, dz \tag{4.112}$$

and rewrite (4.111) after changing the variables in the following simple form:

$$\begin{aligned} K(x(\tau), x'(0)|t) &= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_{A_n} \frac{1}{z-\tau} K\big(x(z), x'(0)|t\big)\, dz \\ &= \sum_{n=-\infty}^{\infty} K_\infty\big(x(\tau + 2\pi n), x'(0)|t\big). \end{aligned} \tag{4.113}$$

The last line is 'the sum over images'. As a result of the summation, the function (4.112) which is not periodic in $\tau$ is converted to a periodic function (4.109).

One can conclude that $K_\infty(x(\tau), x'(0)|t)$ is a heat kernel on an infinitely-sheeted Riemannian surface with the branch point at $\rho = 0$ in coordinates $(\rho, \tau)$, where $0 < \rho < \infty$, $-\infty < \tau < \infty$.

The heat kernel on $C_\beta$ (denoted as $K_\beta$) can be obtained from $K_\infty$ by an analogous procedure, i.e. by a summation over images with the period $\beta$

$$K_\beta\big(x(\tau), x'(0)|t\big) = \sum_{n=-\infty}^{\infty} K_\infty\big(x(\tau + \beta n), x'(0)|t\big). \tag{4.114}$$

By taking into account (4.112) and the known summation formula (4.128), see Exercise 4.9, one arrives at a simple representation

$$K_\beta\big(x(\tau), x'(0)|t\big) = \frac{1}{2i\beta}\int_{A_0} \cot\frac{\pi}{\beta}(z-\tau) K\big(x(z), x'(0)|t\big)\, dz, \tag{4.115}$$

known also as the *Sommerfeld formula*. All results concerning heat kernel expansions on manifolds with conical singularities actually follow from (4.115). The formula can be also derived in a traditional manner by using explicit form of eigenfunctions of the Laplace operator on a cone.

To proceed, we transform $A_0$ in (4.115) to a small circle around the point $z = 0$ and a contour $A'$ which consists of two vertical lines, $(-\pi - i\infty, -\pi + i\infty)$ and $(\pi + i\infty, \pi - i\infty)$, so that

$$\begin{aligned} K_\beta\big(x(\tau), x'(0)|t\big) &= K\big(x(\tau), x'(0)|t\big) \\ &\quad + \frac{1}{2i\beta}\int_{A'} \cot\left(\frac{\pi}{\beta}z\right) K\big(x(z), x'(0)|t\big)\, dz. \end{aligned} \tag{4.116}$$

The presence of a conical singularity influences the last term. Consider the smeared trace $K_\beta(f, L; t)$ (see (4.9)) where $f$ is a test function with a finite integral on $C_\beta$. It is sufficient to assume that $f$ does not depend on $\tau$. Then

$$\begin{aligned} K_\beta(f, L; t) &= t^{-1} a_0(f, L) \\ &\quad + \frac{1}{2i}\int_{A'} \cot\left(\frac{\pi}{\beta}z\right) \int_0^\infty \frac{\exp(-\frac{\rho^2}{t}\sin^2\frac{z}{2})}{4\pi t} f(\rho)\rho\, d\rho\, dz, \end{aligned} \tag{4.117}$$

$$a_0(f, L) = \frac{1}{4\pi}\int_{C_\beta} d^2x \sqrt{g}\, f(x). \tag{4.118}$$

At small $t$ the main contribution to the last integral in the r.h.s. of (4.117) comes from the integration near $\rho = 0$. In this region one can replace $f(\rho)$ by its Taylor series and perform the integration. Up to exponentially small terms this yields

$$K_\beta(f, L; t) \sim t^{-1} a_0(f, L) + a_2(f, L) + \cdots, \tag{4.119}$$

$$a_2(f, L) = \frac{1}{12\gamma}\big(\gamma^2 - 1\big) f(0), \tag{4.120}$$

where $\gamma = \frac{2\pi}{\beta}$. To perform integration in the complex plane we used formula (4.131) from Exercise 4.11. As follows from (4.120), the conical singularity yields a delta-function like contribution. It is interesting to note that for a small angle deficit $|\gamma - 1| \ll 1$ this result can be written as

$$a_2(f, L) \simeq \frac{1}{12\pi}(2\pi - \beta) f(0) = \frac{1}{24\pi} \int d^2x \sqrt{g} R_{\text{sing}}(x) f(x), \qquad (4.121)$$

where $R_{\text{sing}}(x)$ is a distributional curvature of the conical space, see Eq. (1.100). In this limit the heat coefficient can be obtained from formula (4.28) for smooth manifolds if one formally replaces the curvature scalar by the distributional curvature.

**The case of general manifolds with conical singularities** has been discussed earlier in Sect. 1.9. Such manifolds possess internal co-dimension two hypersurfaces $\Sigma$ such that in the vicinity of $\Sigma$ a manifold has the structure $C_\beta \times \Sigma$. We assume that there is a global Killing vector field $\partial_\tau$ on the manifold such that conical singularities are fixed points of $\partial_\tau$. A family of such manifolds which have identical local geometry outside $\Sigma$ but different periodicities in $\tau$ is denoted by $\{\mathscr{M}_\beta\}$. The heat trace asymptotics on $\mathscr{M}_\beta$ have the same form as on the base manifolds with co-dimension 1 defects. The conical singularities produce extra terms in the heat coefficients in a form of local invariant functionals on $\Sigma$. The structure of these terms however is not universal and depends on the type of the operator. This follows already from the analysis of the lowest coefficients such as $a_2(f, L)$. Indeed, for operators $L = \Delta^{(0)}$, $\Delta^{(1/2)}$ and $\Delta^{(1)}$ (see Eqs. (3.8), (3.5)) on a family of manifolds $\mathscr{M}_\beta$ without boundaries

$$a_2(f, L) = (4\pi)^{-n/2} \frac{1}{6} \int_{\mathscr{M}_\beta - \Sigma} d^n x \sqrt{g}\, \text{tr}\big(f(6E + R)\big) + \frac{\pi}{3\gamma} \int_\Sigma d^{n-2} y \sqrt{h} f(y) \big[\sigma_1(\gamma^2 - 1) + \sigma_2(\gamma - 1)\big], \qquad (4.122)$$

where $\sigma_1 = \text{tr}\, \text{I}$ for operators $\Delta^{(0)}$ and $\Delta^{(1)}$, $\sigma_1 = -\frac{1}{2}\text{tr}\, \text{I}$ for operator $\Delta^{(1/2)}$, the constant $\sigma_2 = -12$ for $\Delta^{(1)}$ and vanishes for the other operators. Calculations for the spinor Laplacian can be found in Exercise 4.12. Higher coefficients have a similar structure. For example, contribution from the conical singularities to coefficient $a_4(f, L)$ have a form of integrals over $\Sigma$ of the invariants $R_{ii}$, $R_{ijij}$, and $R$ multiplied by polynomials $(\gamma^p - 1)$ where $p = 1, 2, 4$. The invariants are defined in (1.101).

## 4.8 Literature Remarks

The heat equation (4.1), (4.2) was originally used to describe the heat propagation in various media. The solution (4.3) may represent the evolution of temperature $u(x; t)$ at a given point $x$ over time $t$. The form of the operator $L$ in this case is determined by the thermal conductivity of the media. The heat equation has many other applications ranging from study of the Brownian motion to financial mathematics.

Chapter 4 does not pretend to a complete exposition of the invariance properties and mathematical applications of the heat equation. Its focus is on the results having direct applications to quantum field theory. General sources on the heat kernel expansions are [34, 133, 134, 169]. The style of Ref. [243] is closest to that of this book, though [243] contains more material and is somewhat less pedagogical. There are several commonly used names for the heat coefficients $a_p(f, L)$ in (4.9): the Hadamard-Minakshisundaram-DeWitt-Seeley coefficients, the Seeley-Gilkey coefficients, or the Fock-Schwinger-DeWitt coefficients.

We have not discussed general asymptotic expansions (4.9) for the trace $K(Q, L; t)$ where $Q$ is a partial differential operator. This material can be found in the book by Gilkey [134].

An extensive historical survey of the heat equation and invariance properties can be found in the sources listed above. Here we would like to mention some pioneering papers on spectral functions and their applications in quantum field theory [111, 189, 225]. For an overview of the DeWitt approach to the heat kernel one can consult [26, 37, 77]. Details on the applications of the Gilkey method to local boundary conditions can be found in [52]. Recent advances in the worldline formalism are reported in [29].

In our treatment of singular surfaces (branes) we follow [135, 136]. Pioneering papers on a scattering theory on a wedge and a cone belong to Sommerfeld [234]. An analysis of Laplacians on a cone can be found in more recent mathematical works [43, 63, 83, 167]. The quantum theory near cosmic strings and point-like conical singularities was first considered by Dowker [87, 88], as well as by Deser and Jackiw [75]. For a general form of the heat kernel asymptotic expansions on spaces with conical singularities see [89, 90, 120, 121, 124] and a review in [113].

There are several important topics which have not been mentioned in this Chapter. Result (4.17) for a plane implies that the heat kernels can be introduced for Laplace operators on non-compact manifolds. Well-studied examples are operators on hyperbolic type manifolds discussed in detail in [55] along with a number of physical applications. Homogeneous spaces in general are the case which allows simplified or even exact expressions for the corresponding heat kernels. We do not dwell on this interesting topic because it requires elements of harmonic analysis on homogeneous spaces. This would take us too far from the main subject. An interested reader can find these results in a review article [57]. One of the separate and rather broad subjects is approximate methods of calculation of the heat kernels. The methods depend on the properties of the manifolds and on a physical problem where the heat kernel is applied. Some of these approximations are suggested in [17, 26–28]. Finally, we should mention results aimed to extend the heat kernel technique to theories with broken Lorentz symmetry and non-local operators, see e.g. [198].

Also we have not discussed operators and the heat coefficients associated with higher spin theories. Just to show that properties of higher spin theories may be quite unusual we mention an analysis of massive and massless spin 2 and $3/2$ fields on anti-de Sitter backgrounds [82, 96, 97]. The $a_4$ coefficients for the corresponding operators can be computed in the massive and massless cases and it can be shown that the two sets of coefficients do not coincide in the massless limit.

Recommended Exercises are 4.1, 4.2, and 4.12.

## 4.9 Exercises

**Exercise 4.1** Prove that the heat kernel of the operator $L = -\partial_\mu^2$ on the plane $\mathbb{R}^n$ has the form

$$K(x, y|t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{(x-y)^2}{4t}\right),$$

see Eq. (4.17).

**Exercise 4.2** By explicit summation prove formula (4.16) for heat kernel of the Laplace operator on 3-sphere $S^3$.

**Exercise 4.3** Calculate the volume of an $n$-dimensional unit sphere,

$$\operatorname{vol} S^n = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}. \tag{4.123}$$

**Exercise 4.4** Consider a conformal transformation $L_{\epsilon f} = e^{-\epsilon f} L$ of a Laplace operator $L$ where $f$ is a smooth function on a base manifold $\mathscr{M}$. Prove conformal relation (4.55)

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} a_{n-2}(e^{-2\epsilon f} F, L_{\epsilon f}) = 0,$$

where $F$ is a matrix-valued function and $n$ is the dimensionality of $\mathscr{M}$.

**Exercise 4.5** Let $L$ be an operator of Laplace type and let $Q$ be a matrix valued function (an endomorphism). Suppose that the base manifold is Riemannian, flat, and does not have a boundary. The heat trace (4.6) has the expansion

$$K(Q, L; t) \simeq \sum_{p=0}^{\infty} t^{\frac{p-n}{2}} a_p(Q, L). \tag{4.124}$$

Prove the following expressions for the heat kernel coefficients:

$$a_0(Q, L) = (4\pi)^{-n/2} \int_{\mathscr{M}} d^n x \, \operatorname{tr}(Q), \tag{4.125}$$

$$a_2(Q, L) = (4\pi)^{-n/2} \int_{\mathscr{M}} d^n x \, \operatorname{tr}(QE), \tag{4.126}$$

$$a_4(Q, L) = (4\pi)^{-n/2} \int_{\mathscr{M}} d^n x \, \operatorname{tr}\left(Q\left(\frac{1}{6} E_{;\mu}{}^{\mu} + \frac{1}{2} E^2 + \frac{1}{12} \Omega_{\mu\nu} \Omega^{\mu\nu}\right)\right). \tag{4.127}$$

**Exercise 4.6** Consider conformal variation of the metric $g^{\mu\nu} \to e^{-2\sigma} g^{\mu\nu}$ of a manifold $\mathscr{M}$ with a boundary $\partial\mathscr{M}$. Prove formula (4.90) for conformal variations of the extrinsic curvature of $\partial\mathscr{M}$

$$\delta K_j^j = -\sigma K_j^j - (n-1)\sigma_{;n}.$$

**Exercise 4.7** Suggest a method to derive the coefficients $\beta_2^{D,N}$ which determine the contribution of the extrinsic curvature to the heat coefficient $a_2(f, L)$ on a base manifold with a boundary. Prove Eq. (4.96) without using the example of a two-dimensional disc and a hemisphere.

**Exercise 4.8** Let $\mathscr{M}_+ = \mathscr{M}_-$ be two identical manifolds glued along their common boundary $\partial\mathscr{M}_\pm = \Sigma$ to make a manifold $\mathscr{M} = \mathscr{M}_+ \cup \mathscr{M}_-$ with a codimension one defect on $\Sigma$. Take some functions $f_s$ and $f_a$ on $\mathscr{M}_+$ and make from them symmetric and antisymmetric functions on $\mathscr{M}$: $f_s(x_+) = f_s(x_-)$ and $f_a(x_+) = -f_a(x_-)$, where $x_+ \in \mathscr{M}_+$ and $x_- \in \mathscr{M}_-$ are identical points. Show, that $f_s$ and $f_a$ satisfy the conditions (4.103) and (4.105) on $\mathscr{M}$, provided that $f_s$ and $f_a$ on $\mathscr{M}_+$ satisfy the Robin and Dirichlet boundary conditions, respectively.

**Exercise 4.9** Prove the following summation formulae

$$\sum_{k=-\infty}^{\infty} \frac{1}{z+ka} = \frac{\pi}{a} \cot \frac{\pi z}{a}, \qquad z \neq 0, \pm 1, \pm 2, \ldots, \tag{4.128}$$

$$\sum_{k=-\infty}^{\infty} \frac{e^{ik\alpha}}{z+ka} = \frac{\pi}{a} \frac{e^{i(\pi-\alpha)z/a}}{\sin \frac{\pi z}{a}}, \qquad z \neq 0, \pm 1, \pm 2, \ldots, \tag{4.129}$$

where $a$ is real and $\alpha$ lies in the interval $0 < \alpha < 2\pi$.

**Exercise 4.10** Consider the heat kernel for the scalar Laplacian on a cone $C_\beta$ with the following periodicity property

$$K_{\beta,\alpha}\big(x(\tau+\beta), x'(0)|t\big) = e^{i\alpha} K_{\beta,\alpha}\big(x(\tau), x'(0)|t\big), \tag{4.130}$$

for $0 < \alpha < 2\pi$. Find a generalization of the Sommerfeld formula (4.115) which relates $K_{\beta,\alpha}$ with the heat kernel on the plane.

**Exercise 4.11** Prove the following formulae

$$\frac{1}{i\beta} \int_{A'} dz \cot \frac{\pi}{\beta} z \frac{1}{\sin^2 \frac{z}{2}} = -\frac{2}{3}(\gamma^2 - 1), \tag{4.131}$$

$$\frac{1}{i\beta} \int_{A'} \frac{dz}{\sin \frac{\pi}{\beta} z} \frac{\cos \frac{z}{2}}{\sin^2 \frac{z}{2}} = -\frac{1}{3}(\gamma^2 - 1), \tag{4.132}$$

where $\gamma = \frac{2\pi}{\beta}$.

**Exercise 4.12** Consider on $C_\beta$ the spinor Laplacian $\Delta^{(1/2)} = -\nabla^\mu \nabla_\mu$, see Eq. (3.8). The covariant derivatives $\nabla_\mu = \partial_\mu + w_\mu$ can be determined by vielbeins in the polar coordinates $\rho, \tau$. In the basis $\gamma_\mu = (\sigma_1, \sigma_2)$, where $\sigma_k$ are the Pauli matrices the Levi-Civita connection has a single non-vanishing component $w_\tau = -\frac{i}{2}\sigma_3$.

Find an analog of the Sommerfeld representation (4.115) which relates the heat kernel of the spinor Laplacian on $C_\beta$ to the heat kernel of the scalar Laplacian on the plane. Calculate on $C_\beta$ the corresponding spinor heat coefficient $a_2(f, L)$.

# Chapter 5
# Spectral Functions

## 5.1 Where do the Spectral Functions Come From?

For a finite-dimensional linear operator (a matrix) one can define a determinant, a trace, and traces of powers of the operator. All these objects are independent of a particular orthonormal basis chosen to represent the operator, and therefore they contain important invariant information. One can show that notions of a determinant can be extended to infinite-dimensional operators, and the difficulties with the convergence of series, which naively define determinants and traces, can be resolved. A systematic approach to the invariants associated with linear operators is based on the notion of *spectral functions*. These are functions defined on the spectrum of the operator which depend additionally on a complex or real parameter. A typical example of a spectral function is the heat trace $K(L;t)$ for a Laplace operator $L$, see Eq. (4.8).

To illustrate how spectral functions appear in the context of quantum physics let us return to the computation of the vacuum energy discussed in Sect. 2.5. The vacuum energy for a real field is formally given by the series, see (2.49),

$$E = \frac{1}{2}\sum_i \omega_i, \tag{5.1}$$

over the frequencies $\omega_i$ of single-particle modes. Formula (5.1) does not make much sense since $\omega_i$ grow with $i$, and the sum in (5.1) is divergent. As has been already explained, a sum like (5.1) should imply a regularization at large frequencies. Let us consider a particular method to regularize the series. Basing on our finite-dimensional experience only, we can rewrite (5.1) formally as

$$E = \frac{1}{2}\,\mathrm{Tr}(H), \tag{5.2}$$

which is still ill-defined as well. Then by introducing a regularization parameter $s$ Eq. (5.2) is replaced by

$$E_s = \frac{1}{2}\sum_i \omega_i^{1-s} = \frac{1}{2}\,\mathrm{Tr}(H^{1-s}). \tag{5.3}$$

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For a sufficiently large positive $s$ the sum in (5.3) is convergent. One can make calculations for such values of $s$ and then continue $E_s$ analytically to the "physical" value $s=0$. By itself this procedure still does not make the result finite, but it allows to isolate a divergent part and to remove it eventually by a suitable renormalization procedure.

We have just demonstrated, how one can define a spectral function $E_s$ of the operator $H$ depending on a spectral parameter $s$. A natural extension of this construction is the so-called generalized zeta-function of the operator, which will be considered below in detail.

## 5.2 The Riemann Zeta-Function

The zeta-function of differential operators was first introduced as a generalization of the Riemann zeta-function. That is why we consider the Riemann zeta-function first. This function is defined as a sum over natural numbers rather than over a spectrum of a differential operator. Nevertheless, the analytical properties of all zeta-functions are very similar.

Consider the series

$$\zeta_R(s,a)=\sum_{n=0}^{\infty}(n+a)^{-s}, \tag{5.4}$$

where $\mathrm{Re}\, s>1$ and $a\neq 0,-1,-2,\ldots$. Function $\zeta_R(s,a)$ is called the generalized Riemann zeta-function, see [30], or the Hurwitz zeta-function.

The Riemann zeta-function $\zeta(s)$ is a particular case of (5.4)

$$\zeta_R(s)=\zeta_R(s,1)=\sum_{n=1}^{\infty}n^{-s}. \tag{5.5}$$

The single-particle energies of the simplest scalar field model on a circle, discussed in Exercise 2.13, are $\omega_n\propto n$, $n=1,2,\ldots$ and the regularized vacuum energy (5.3) for this model is expressed in terms of the zeta-function, $E_s\propto\zeta_R(s)$.

In practical calculations it is convenient to use integral representations for the Hurwitz functions. To find such a representation for $\zeta_R(s,a)$ note that for $\Re\, s>0$, $\Re\, a>0$

$$\int_0^\infty dt\, e^{-(n+a)t}t^{s-1}=(n+a)^{-s}\Gamma(s), \tag{5.6}$$

where $\Gamma(s)$ is the gamma-function,

$$\Gamma(s)=\int_0^\infty dt\, e^{-t}t^{s-1}. \tag{5.7}$$

It follows from (5.7) that

$$\zeta_R(s,a)=\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\frac{e^{(1-a)t}}{e^t-1}. \tag{5.8}$$

Formula (5.8) can be used to construct an analytical continuation of $\zeta_R(s,a)$ to the whole complex plane. Furthermore, it can be shown that $\zeta_R(s,a)$ has only one singular point $s=1$ where it has a simple pole. This singularity appears as a result of a divergence in the integral in (5.8) at the lower limit of integration. By using (5.8) it is easy to show that near $s=1$

$$\zeta_R(s,a) \simeq \frac{1}{s-1}. \tag{5.9}$$

To study the analytical structure of zeta-functions we need the Bernoulli polynomials $B_n(x)$, which are defined as coefficients in the series

$$se^{xs}(e^s-1)^{-1} = \sum_{n=0}^{\infty} B_n(x)\frac{s^n}{n!}, \quad |s|<2\pi. \tag{5.10}$$

It is instructive to introduce also the Bernoulli numbers $B_n$

$$s(e^s-1)^{-1} = \sum_{n=0}^{\infty} B_n \frac{s^n}{n!}, \quad |s|<2\pi. \tag{5.11}$$

The Bernoulli polynomials are expressed in terms of Bernoulli numbers as

$$B_n(x) = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} B_m x^{n-m}, \tag{5.12}$$

$$B_n(0) = B_n. \tag{5.13}$$

One has

$$B_0(x)=1, \quad B_1(x)=x-\frac{1}{2}, \quad B_2(x)=x^2-x+\frac{1}{6}, \tag{5.14}$$

$$B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x, \quad B_4(x)=x^4-2x^3+x^2-\frac{1}{30}, \quad \ldots \tag{5.15}$$

Note that

$$B_{2n+1}=0 \tag{5.16}$$

for $n \geq 1$. The Bernoulli polynomials can be used to find values of the zeta-function at negative integer arguments. To this aim let us represent (5.8) as

$$\zeta_R(s,a) = \frac{1}{\Gamma(s)} \int_0^1 dt\, t^{s-1} \frac{e^{(1-a)t}}{e^t-1} + I_1(s,a), \tag{5.17}$$

where $I_1(s,a)$ is determined by the integral over $t$ from 1 to infinity. We can use the expansion (5.11) in the integral in the first term on the right hand side of (5.17) to write

$$\zeta_R(s,a) = \frac{1}{\Gamma(s)} \sum_{n=0}^{N} \frac{B_n(1-a)}{n!(s+n-1)} + I_1(s,a) + I_2(s,a), \tag{5.18}$$

where $N$ is a natural number such that $N-1 > -\Re s$. The term $I_2(s,a)$ is equal to

$$I_2(s,a) = \frac{1}{\Gamma(s)} \int_0^1 dt\, t^{s-2} \left[ t \frac{e^{(1-a)t}}{e^t - 1} - \sum_{n=0}^{N} B_n(1-a) \frac{s^n}{n!} \right]. \tag{5.19}$$

It is easy to see that both $I_1(s,a)$, $I_2(s,a)$ vanish for $s = -k$ ($k < N-1$) because of the poles of the gamma-function at those points,

$$\Gamma(s) \simeq (-1)^k \frac{1}{k!(s+k)}.$$

There is however a non-trivial contribution at $s = -k$ which comes from the first term in (5.18) where the zero of the gamma-function in the denominator is compensated. As a result, we find

$$\zeta_R(-k,a) = (-1)^k \frac{B_{k+1}(1-a)}{k+1} = -\frac{B_{k+1}(a)}{k+1}. \tag{5.20}$$

Here we have used the property $B_n(1-x) = (-1)^n B_n(x)$. With the help of (5.20) one can also find the values of the Riemann zeta-function at *positive* integers. The key formula here is the Hurwitz relation

$$\zeta_R(s) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \sin\frac{\pi s}{2} \zeta_R(1-s). \tag{5.21}$$

This yields

$$\zeta_R(2m) = (-1)^{m+1} (2\pi)^{2m} \frac{B_{2m}}{2(2m)!}. \tag{5.22}$$

Note, that because of (5.16), the Riemann zeta-function has zeros at all negative integers.

## 5.3 Zeta-Function of an Operator

One can generalize the notion of the Riemann zeta-function and introduce a zeta-function ($\zeta$-function) of an operator. Let $L$ be a self-adjoint second order elliptic differential operator. Suppose that the eigenvalues $\lambda$ of $L$ are real and positive, i.e. $L$ is a *positive-definite* operator. In this case the $\zeta$-function of the operator $L$ is defined as

$$\zeta(s; L) = \sum_{\lambda} \lambda^{-s}. \tag{5.23}$$

In Chap. 3 we argued that on any compact manifold the large eigenvalue asymptotics of all second-order elliptic operators look similarly (up to a factor). Therefore, one can easily estimate the growth $\lambda^{-s}$ at large eigenvalues and conclude that series (5.23) converges if $\Re s > n/2$, where $n$ is the dimensionality of the base manifold $\mathscr{M}$, see Exercise 5.1.

The $\zeta$-function can be analytically continued in the parameter $s$ from the domain $\Re s > n/2$ to the entire complex plane. One can show that it is a meromorphic function of $s$ which has a finite number of poles on the real axis $\Im s = 0$. The poles can be found with the help of asymptotic expansion of the heat kernel (4.9). To this aim we express the $\zeta$-function in terms of the corresponding heat trace $K(L;t)$

$$\zeta(s;L) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} K(L;t). \tag{5.24}$$

The above formula follows from the integral representation for the $\Gamma$-function (5.7). At large $t$ the heat trace behaves as $K(L;t) \sim e^{-t\lambda_0}$, where $\lambda_0$ is the lowest eigenvalue of $L$. Since the definition of the zeta-function is valid for positive operators only, $\lambda_0 > 0$, and the integral (5.24) is convergent at the upper limit. The divergences which may result in poles come from the lower integration limit. We represent

$$\zeta(s;L) = f_1(s) + f_2(s), \tag{5.25}$$

$$f_1(s,L) = \frac{1}{\Gamma(s)} \int_0^1 dt\, t^{s-1} K(t;L). \tag{5.26}$$

The function $f_2(s)$ has no poles. For $f_1$ we can replace $K(t;L)$ by its asymptotic (4.9) (with the convention $a_p(L) \equiv a_p(1,L)$). This yields

$$f_1(s) \sim \frac{1}{\Gamma(s)} \sum_{p=0}^{\infty} \frac{a_p(L)}{s + \frac{p-n}{2}}. \tag{5.27}$$

This shows, that the function $\Gamma(s)\zeta(s;L)$ has simple poles at $s = \frac{n-p}{2}$ with the residues

$$\mathrm{Res}(\Gamma(s)\zeta(s;L))_{s=\frac{n-p}{2}} = a_p(L). \tag{5.28}$$

Since the product $\Gamma(s)\zeta(s;L)$ has a simple pole at $s = 0$, the $\zeta$-function itself is regular at $s = 0$. Consequently, the derivative $\zeta'(s;L)$ also is well defined at $s = 0$. Another consequence of (5.25), (5.27) is the expression

$$\zeta(-k;L) = (-1)^k k! a_{n+2k}(L), \tag{5.29}$$

where $k$ is a natural number.

These results can be extended with corresponding modifications to positive *semi-definite* operators, i.e., self-adjoint operators whose eigenvalues are not negative. The generalized $\zeta$-function of such an operator $L$ is defined as

$$\zeta(s;L) = \sum_{\lambda \neq 0} \lambda^{-s}. \tag{5.30}$$

Let $N_0$ be the number of modes corresponding to vanishing eigenvalue $\lambda = 0$ (zero modes). We define

$$\tilde{K}(L;t) = K(L;t) - N_0. \tag{5.31}$$

Then

$$\zeta(s;L) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \tilde{K}(L;t). \tag{5.32}$$

If $a_p$ are the coefficients of the asymptotic expansion of $K(L;t)$, the coefficients of $\tilde{K}(L;t)$ are $\tilde{a}_p$, where $\tilde{a}_p = a_p$ if $p \neq n$ and $\tilde{a}_n = a_n - N_0$. The residues of $\Gamma(s)\zeta(s,L)$ are determined by $\tilde{a}_p$ through formula (5.28).

Clearly, a finite number of negative modes is not a problem since the trace over the negative subspace is finite-dimensional. Also, the eigenvalues may be complex in a "controllable way". In this case, one should define the phase of $\lambda^{-s}$, i.e. to place a branch cut for $\ln\lambda$ in the complex plane. To do this properly one usually requires that the leading symbol of $L$ has no spectrum in conical neighborhood of a ray coming from the origin.

If $L$ is a positive operator and $Q$ is some other operator then in addition to $\zeta(s;L)$ one can define the function

$$\zeta(s;L,Q) = \mathrm{Tr}(QL^{-s}). \tag{5.33}$$

Our final remark concerns extension of these notions to arbitrary self-adjoint operators which have finite or infinite (like in case of the Dirac operator) number of negative eigenvalues. For such an operator $L$ one can introduce the so-called eta-function

$$\eta(s;L) = \sum_{\lambda>0} \lambda^{-s} - \sum_{\lambda<0} |\lambda|^{-s}. \tag{5.34}$$

If there are no vanishing eigenvalues one can write (5.34) as

$$\eta(s;L) = \mathrm{Tr}(L(L^2)^{-\frac{1}{2}(s+1)}). \tag{5.35}$$

The latter definition enables one to relate the $\eta$-function with the $\zeta$-function (5.33)

$$\eta(s;L) = \zeta\left(\frac{1}{2}(s+1), L, L^2\right). \tag{5.36}$$

The eta-function characterizes the spectral asymmetry of operator $L$.

## 5.4 Spectral Density and Its Asymptotic Properties

In this section we introduce other types of spectral functions. Let $L$ be a second order self-adjoint positive-definite elliptic differential operator acting on the sections of a bundle over a compact base manifold $\mathscr{M}$. One can define the *spectral density* of $L$

$$\rho(\lambda) = \sum_{\lambda_0}^{\infty} \delta(\lambda_k - \lambda), \tag{5.37}$$

and the *counting function*

$$N(\lambda) = \int_0^\lambda d\sigma\, \rho(\sigma). \tag{5.38}$$

Here $\lambda$ is a real parameter and the sum in (5.37) goes over all eigenvalues $\lambda_k$ of $L$.

The counting function yields the total number of eigenvalues $\lambda_k$ which do not exceed $\lambda$. Strictly speaking, $N(\lambda)$ may be approximated by a smooth function only at large $\lambda$. In this limit all counting functions exhibit universal asymptotic properties determined by the Weyl formula

$$N(\lambda) \simeq \frac{\lambda^{n/2} a_0}{\Gamma(n/2+1)} \sim \lambda^{n/2} V_n, \tag{5.39}$$

where $a_0$ is the leading heat kernel coefficient in (4.9), and $V_n$ is the volume of $\mathscr{M}$. Before we consider derivation of (5.39) let us note that computation of the sub-leading terms in the Weyl formula meets a difficulty because starting with a certain order these terms become smaller then fluctuations of $N(\lambda)$ when $\lambda$ goes from one eigenvalue to the next one. This means, that an expansion in powers of $\lambda$ cannot approximate $N(\lambda)$.

The way out of this difficulty is to work with smoothed functions $N(\lambda)$ and $\rho(\lambda)$. The smoothing can be done in different ways, and one of them is to use the so-called Riesz means. A "smoothed" spectral function, $\rho_\alpha(\lambda)$, is defined as

$$\rho_\alpha(\lambda) = \frac{1}{\Gamma(\alpha)} \int_0^\lambda (\lambda - \sigma)^{\alpha-1} \rho(\sigma)\, d\sigma, \tag{5.40}$$

where $\alpha$ is a complex parameter, $\Re\,\alpha > 0$. It follows from (5.40) that $\rho_1(\lambda) = N(\lambda)$ and $\rho_0(\lambda) = \rho(\lambda)$. Note also that $\rho_\alpha(\lambda)$ can be interpreted as a fractional derivative $\partial_\lambda^{-\alpha} \rho(\lambda)$, see [202].

There is a simple relation between $\rho_\alpha(\lambda)$ and the trace of the heat kernel,

$$\int_0^\infty e^{-t\lambda} \rho_\alpha(\lambda)\, d\lambda = t^{-\alpha} K(L;t). \tag{5.41}$$

Suppose now, that $\alpha \neq k/2$ where $k$ is an integer. Then, at large $\lambda$ the spectral function $\rho_\alpha(\lambda)$ is represented by the asymptotic series

$$\rho_\alpha(\lambda) \sim \sum_{p=0}^\infty a_p \frac{\lambda^{(n-p)/2+\alpha-1}}{\Gamma(\frac{n-p}{2}+\alpha)}. \tag{5.42}$$

One can substitute (5.42) in (5.41), and check that this series reproduces the short $t$ expansion (4.9).

In the limit $\alpha \to 1$ the leading term in (5.42) correctly reproduces (5.39). When regularization is removed one encounters the above mentioned difficulty with the sub-leading terms in the Weyl formula because all terms in (5.42) with $p = n + 2N$, where $N$ is a natural number, disappear at $\alpha = 1$. The expansion terminates and cannot reproduce the entire heat kernel asymptotics (4.9).

The problem can be formally avoided if the sub-leading terms are treated as generalized functions. One can write

$$\rho(\lambda) \sim \sum_{p=0}^{\infty} a_p \partial_\lambda^{\frac{p-n}{2}} \delta(\lambda). \tag{5.43}$$

We used the fact that [128]

$$\lim_{\beta \to -n} \frac{x_+^{\beta-1}}{\Gamma(\beta)} = \partial_x^n \delta(x), \tag{5.44}$$

where $n = 0, 1, 2, \ldots$ and $x_+^{\beta-1} = x^{\beta-1}$ for $x \geq 0$ and $x_+^{\beta-1} = 0$ for $x < 0$. We shall use formula (5.43) in Chap. 6.

## 5.5 Determinants of Second Order Elliptic Operators

Consider a finite-dimensional non-degenerate matrix $L$ with positive eigenvalues $\lambda$. For each $\lambda$ one can write the identity $\ln\lambda = -d(\lambda^{-s})/ds|_{s=0}$ and define the determinant of $L$ by the relation

$$\ln\det L = -\frac{d}{ds}\,\mathrm{Tr}(L^{-s})|_{s=0}. \tag{5.45}$$

If $L$ is a differential operator the sum $\sum_\lambda \ln\lambda$ is divergent. Ray and Singer proposed to use (5.45) and formula (5.23) to define the determinant of an operator in terms of its zeta-function. If $L$ is a positive-definite self-adjoint second order operator one can give the following definition

$$\ln\det L \equiv -\zeta'(0; L). \tag{5.46}$$

As was explained in Sect. 5.3, the derivative $\zeta'(0; L)$ is well defined because the zeta-function is regular at $s = 0$ .

The Ray-Singer (5.46) definition omits all divergences present in the sum $\sum_\lambda \ln\lambda$. The structure of these divergences however is of interest for physical applications. It is time to investigate this question in more detail. Let us start with the asymptotic formula

$$-\int_\delta^\infty \frac{dt}{t} e^{-t\lambda} = \ln\lambda\delta + \gamma + O(\lambda\delta), \tag{5.47}$$

where $\delta > 0$, $\gamma_E = 0.577216\ldots$ is the Euler constant. One can use (5.47) as a motivation for the following definition:

$$\sum_\lambda \ln\lambda \equiv -\int_\delta^\infty \frac{dt}{t}\,\mathrm{Tr}(e^{-tL}). \tag{5.48}$$

The next step is to shift in (5.48) the power $t^{-1}$ to $t^{1-s}$, put $\delta = 0$ and define a *regularized* determinant as

$$(\ln\det L)_s = -\mu^{2s} \int_0^\infty \frac{dt}{t^{1-s}} K(L; t). \tag{5.49}$$

A constant $\mu$ of dimension of the mass was introduced here to keep proper dimensionality of the whole expression. The right hand side of (5.49) is convergent if $s > n/2$ and one can use Eq. (5.24) to obtain

$$(\ln \det L)_s = -\mu^{2s}\Gamma(s)\zeta(s; L). \tag{5.50}$$

This is so-called *zeta-function regularized* determinant. Near the value $s = 0$ this expression is divergent and reads

$$(\ln \det L)_s = -\left(\frac{1}{s} - \gamma_E + \ln \mu^2\right)\zeta(0, L) - \zeta'(0, L) + \mathscr{O}(s). \tag{5.51}$$

One can identify the determinant with the finite part of $(\ln \det L)_s$

$$\ln \det L \equiv -\zeta'(0, L) - \ln(\mu^2) \cdot \zeta(0, L). \tag{5.52}$$

By simply neglecting the second term on the right hand side of (5.52), which is equivalent to the choice $\mu = 1$, one reproduces the Ray-Singer definition (5.46). It follows from (5.29), that

$$\zeta(0, L) = a_n(L), \tag{5.53}$$

therefore the divergent part of the zeta-regularized determinant is determined by the heat kernel coefficient $a_n(L)$.

These definitions may be extended to higher order positive elliptic operators. Suppose now, that the operators $A$, $B$ and $AB$ have zeta-determinants. In contrast to the finite-dimensional case, determinant of the product need not be the product of determinants. The fraction

$$\frac{\det A \cdot \det B}{\det(AB)} \tag{5.54}$$

is called therefore the *multiplicative anomaly*.

We conclude this section by a somewhat disappointing remark: there is no universal definition of the determinant valid for any elliptic differential operator, see Sect. 5.10.

## 5.6 Zeta-Function and Determinant of the Dirac Operator

The Dirac operator has an infinite number of negative modes and the definition of its regularized determinant requires some modifications with respect to what has been discussed in the previous section. Suppose that the Dirac operator $\not{D}$ is selfadjoint, and, therefore, that the spectrum is real. Let us assume for a while that the spectrum of eigenvalues is discrete $\lambda_k$ and does not contain zero modes. The corresponding zeta-function may be defined as above,

$$\zeta(s, \not{D}) = \sum_{\lambda_k} \lambda_k^{-s}. \tag{5.55}$$

The new feature here is the presence of negative modes and a related ambiguity in the phase of their contributions. We fix this ambiguity in the following way:

$$\zeta(s, \not{D}) = \sum_{\lambda_k > 0} \lambda_k^{-s} + e^{-i\pi s} \sum_{\lambda_k < 0} (-\lambda_k)^{-s}. \tag{5.56}$$

In any even number of dimensions, there is a chirality matrix $\gamma_*$. For some choices of the Dirac operator (see, e.g., Eq. (8.17) below) the chirality matrix anti-commutes with $\not{D}$, see Sect. 5.9 for a more detailed discussion. Consequently, the non-zero spectrum of $\not{D}$ is symmetric with respect to the reflection $\lambda \to -\lambda$. Therefore, one can write

$$\begin{aligned}\zeta(s, \not{D}) &= (1 + e^{-i\pi s}) \sum_{\lambda_k > 0} \lambda_k^{-s} = (1 + e^{-i\pi s}) \sum_{\lambda_k > 0} (\lambda_k^2)^{-s/2} \\ &= \frac{1}{2}(1 + e^{-i\pi s}) \sum_{\lambda_k \neq 0} (\lambda_k^2)^{-s/2} = \frac{1}{2}(1 + e^{-i\pi s})\zeta(s/2, \not{D}^2).\end{aligned} \tag{5.57}$$

The zeta-regularized determinant of the Dirac operator is defined as in (5.50)

$$\ln(\det \not{D})_s = -\mu^s \Gamma(s) \zeta(s, \not{D}). \tag{5.58}$$

Note that the factor $\mu^s$ (instead of $\mu^{2s}$) appears in (5.58) due to a different canonical dimension of the Dirac operator. Near $s = 0$ we have

$$\ln(\det \not{D})_s = -\left(\frac{1}{s} - \gamma_E + \ln \mu - \frac{i\pi}{2}\right)\zeta(0, \not{D}^2) - \frac{1}{2}\zeta'(0, \not{D}^2). \tag{5.59}$$

This definition is a useful tool to study properties of the determinant of the Dirac operator. We shall apply it to the analysis of the anomalous behavior of regularized determinants under symmetry transformations.

## 5.7 Transformations of Determinants of Laplace Type Operators

In a number of applications one needs to know how determinants of operators transform under variations of background fields. An important class of these transformations is related to symmetries of the classical action. This class will be discussed in Chap. 8 in detail. In this section, we derive some useful general relations.

Consider two Laplace type operators, $L$ and $\bar{L}$, which are related as $L = e^{\frac{1}{2}\mathcal{O}} \bar{L} e^{\frac{1}{2}\mathcal{O}}$, and connect them through a homotopy $L(\alpha) = e^{\frac{\alpha}{2}\mathcal{O}} \bar{L} e^{\frac{\alpha}{2}\mathcal{O}}$ with $\alpha \in [0, 1]$. We further assume that all $L(\alpha)$ are of Laplace type, and that $e^{\frac{\alpha}{2}\mathcal{O}}$ form a one-parameter group. This implies in particular that $e^{\frac{\alpha}{2}\mathcal{O}}$ is invertible for all $\alpha$.

Let us make a short pause to explain a subtlety regarding infinite-dimensional operators. In the finite-dimensional case, any operator of the form $e^A$ is invertible with the inverse being $e^{-A}$. In the infinite-dimensional case this is not necessarily so. For example, $e^{-t\Delta}$ with the standard Laplacian $\Delta$ and $t > 0$ is *not* invertible since $e^{t\Delta}$ grows exponentially at large momenta and is not defined on a dense domain of $L^2$. Therefore, the operators $e^{-t\Delta}$ do not form a group, but only a semi-group, called the heat semi-group.

Returning to the problem in question, we may write

$$L(\alpha+\delta\alpha) = L(\alpha) + \delta\alpha\frac{1}{2}(\mathscr{O}L(\alpha) + L(\alpha)\mathscr{O}) \tag{5.60}$$

to the linear order in $\delta\alpha$. To calculate variation of the heat kernel, it is convenient to use the Duhamel formula

$$e^{A+B} = e^A + \int_0^1 e^{(A+B)u} B e^{(1-u)A}\, du, \tag{5.61}$$

which has a purely combinatoric origin. With the help of this relation we can write

$$\delta e^{-tL} = -\int_0^t du\, e^{-uL}(\delta L)e^{-(t-u)L}. \tag{5.62}$$

The formula is valid if the operators under the integral exist, which is indeed our case since both $u$ and $t-u$ remain non-negative in the whole interval of integration. By taking the trace, we obtain

$$\frac{d}{d\alpha}K(1,L(\alpha),t) = -t\,\mathrm{Tr}(\mathscr{O}L(\alpha)e^{-tL(\alpha)}) = t\frac{d}{dt}K(\mathscr{O},L(\alpha),t). \tag{5.63}$$

Equation (5.63) can be written also in an infinitesimal form

$$\delta\,\mathrm{Tr}\,e^{-tL(\alpha)} = -t\,\mathrm{Tr}\big(\mathscr{O}(\alpha)L(\alpha)e^{-tL(\alpha)}\big)\delta\alpha. \tag{5.64}$$

Apart from the variations (5.60) generated by the operator $\mathscr{O}$, some other, rather complicated transformations of $L$ may lead to the same result (5.64).

Next, let us analyze the variation of zeta-function. Since $L(\alpha)$ can have zero modes, we write

$$\zeta(s,L,\mathscr{O}) \equiv \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\big[\mathrm{Tr}\big(\mathscr{O}e^{-tL} - \mathrm{Pr}_N\mathscr{O}\big)\big] \tag{5.65}$$

(with $\mathrm{Pr}_N$ being a projector on the null-subspace of $L$) and consider variation of the unsmeared zeta function

$$\zeta(s,L(\alpha)) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\big(\mathrm{Tr}\,e^{-tL_\alpha} - N\big), \tag{5.66}$$

where $N$ is the number of zero modes. Since the transformations $e^{\frac{1}{2}\alpha\mathscr{O}}$ are invertible for all $\alpha$, the number of zero modes does not change, and one obtains:

$$\frac{d}{d\alpha}\zeta(s,L(\alpha)) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\,\mathrm{Tr}\big(-t\mathscr{O}L(\alpha)e^{-tL(\alpha)}\big). \tag{5.67}$$

Let us look more attentively at the operator under the trace. On the zero subspace of $L(\alpha)$ the heat operator equals to the identity, and the operator $L(\alpha)$ maps this subspace to zero. Therefore, nothing changes in the equation above if we replace

$e^{-tL(\alpha)}$ by $(e^{-tL(\alpha)} - \mathrm{Pr}_N)$. After integrating over $t$ one gets back the zeta-function, i.e.

$$\frac{d}{d\alpha}\zeta(s, L(\alpha)) = -\frac{\Gamma(1+s)}{\Gamma(s)}\,\mathrm{Tr}\big(\mathscr{O}L(\alpha)L(\alpha)^{-(s+1)}\big) = -s\zeta(s, L(\alpha), \mathscr{O}). \tag{5.68}$$

This relation may now be taken at $s = 0$. One immediately concludes, that $\zeta(0, L(\alpha))$ does not depend on $\alpha$, while the variation of the derivative $\zeta'(0, L(\alpha))$ is proportional to $\zeta(0, L(\alpha), \mathscr{O})$,

$$\zeta(0, L) = \zeta(0, \bar{L}), \tag{5.69}$$

$$\zeta'(0, L) = \zeta'(0, \bar{L}) - \int_0^1 d\alpha\, \zeta(0, L(\alpha), \mathscr{O}), \tag{5.70}$$

where, we remind that $\bar{L} = L(0)$ and $L = L(1)$.

The last formula defines also the transformation rule of Ray-Singer determinant (5.46)

$$\delta \ln \det L = \zeta(0, L, \mathscr{O})\delta\alpha = (a_n(\mathscr{O}, L) - \mathrm{Pr}_N(\mathscr{O}))\delta\alpha, \tag{5.71}$$

given here in the infinitesimal form. $\mathrm{Pr}_N(\mathscr{O})$ denotes here the trace of $\mathscr{O}$ restricted to the zero subspace of $L$.

## 5.8 Other Definitions of Determinants

There are other regularizations of determinants besides the zeta-function one. Restrict, for example, the integration in (5.49) at some small cutoff parameter $\delta > 0$. Since the integral is now convergent, one can take the limit $s \to 0$ and define the regularized determinant by formula

$$(\ln \det L)_{\mathrm{PTC},\delta} = -\int_\delta^\infty \frac{dt}{t}\tilde{K}(L; t). \tag{5.72}$$

This regularization is known as the *proper-time cutoff* (*PTC*) *regularization*. In general, such a regularization can be applied to any spectral function associated to the operator $L$. In particular, one can consider a "regularized zeta-function",

$$\zeta_\delta(s; L) = \frac{1}{\Gamma(s)}\int_\delta^\infty dt\, t^{s-1}\tilde{K}(L; t). \tag{5.73}$$

The difference between $\zeta_\delta(s; L)$ and $\zeta(s; L)$, Eq. (5.32), is that the regularized zeta-function vanishes at $s = 0$, $\zeta_\delta(0; L) = 0$.

In the limit $\delta \to 0$ determinant (5.72) is divergent,

$$(\ln \det L)_{\mathrm{PTC},\delta} \sim \sum_{p=0}^{n-1} \frac{2a_p(L)}{p-n}\delta^{\frac{p-n}{2}} + (a_n(L) - N)\ln\delta. \tag{5.74}$$

This is a complete structure of divergences of a determinant of a second order operator. One can note, by comparing (5.51) with (5.74), that the zeta-function method reproduces the logarithmic divergences only. In the framework of the PTC regularization the finite (renormalized) part of the determinant can be defined as

$$\ln\det L = \lim_{\delta\to 0}\left[(\ln\det L)_{\mathrm{PTC},\delta} - \sum_{p=0}^{n-1}\frac{2a_p(L)}{p-n}\delta^{\frac{p-n}{2}} - \zeta(0,L)\ln\delta\right]. \quad (5.75)$$

There are other definitions of determinants which we do not discuss here. All of them are based on certain regularization prescriptions followed by a subtraction of the divergent parts. The value of the finite part depends, of course, on the subtraction. We remark that it is subtraction procedure (5.75) which results in the same transformations of the determinants as considered in Sect. 5.7 (though the Ray-Singer definition appears to be the most suitable to derive these transformations). Consider, as an example, the simplest transformation of the operator $\delta L = L\alpha$, where $|\alpha| \ll 1$. This is just a rescaling, $\bar{L} = \mu L$, with $\mu = 1 + \alpha$. From (5.71) one immediately gets

$$\delta \ln\det L = \zeta(0,L)\alpha \quad (5.76)$$

for the Ray-Singer definition. Let us show that this transformation is reproduced by the PTC definition (5.75). First, note that the regularized quantity $(\ln\det L)_{\mathrm{PTC},\delta}$ does not change if together with rescaling of the operators we rescale the cutoff parameter, $\bar{\delta} = \mu^{-1}\delta$. Consequently, it follows from definition (5.75) that

$$\delta\ln\det L = \lim_{\delta\to 0}\left[\sum_{p=0}^{n-1}\left(\frac{2a_p(L)}{p-n}\delta^{\frac{p-n}{2}} - \frac{2a_p(\bar{L})}{p-n}\bar{\delta}^{\frac{p-n}{2}}\right) + \zeta(0,L)\ln\delta/\bar{\delta}\right], \quad (5.77)$$

where we took into account that $\zeta(0,\bar{L}) = \zeta(0,L)$. According to results of Sect. 4.4 canonical mass dimension of $a_p(\bar{L})$ is $p-n$ while the dimension of $\delta$ is 2. Therefore, the terms in (5.77) with powers of $\delta$ cancel out, while the last term reproduces (5.76).

## 5.9 Index Theory

There is an important connection between spectral properties of differential operators and topology. This connection is the subject of the index theory which we briefly outline in the present section. The idea is quite simple and is based on the index of an operator which is the difference between the number of zero modes of the operator and the number zero modes of its adjoint. The index can be related to an integral of some local invariants which are combinations of heat kernel coefficients. Since the index is an integer it cannot change under smooth deformations of the metric of the base manifold or of a bundle over the manifold. This means, that the index is a topological invariant. By studying index of operators one gets not only topological invariants, but also very convenient local expressions for them.

Let $\mathscr{M}$ be a compact Riemannian manifold with or without boundary and $\mathscr{E}_1$, $\mathscr{E}_2$ be two vector bundles over $\mathscr{M}$. Each $\mathscr{E}_k$ is supposed to be equipped with a positive definite inner product $(.,.)_k$, $k = 1,2$, see (3.13). Consider an operator $D_+$ which acts on sections of $\mathscr{E}_1$ and maps them to sections of $\mathscr{E}_2$. By using the inner products we can define an adjoint to $D_+$ by requiring $(\varphi_2, D_+\varphi_1)_2 = (D_+^\dagger \varphi_2, \varphi_1)_1$ for all sufficiently well behaving sections $\varphi_k$ of $\mathscr{E}_k$. To make the notations more symmetric we denote $D_- \equiv D_+^\dagger$. Let us define operators $L_1 = D_- D_+$ and $L_2 = D_+ D_-$ acting on smooth sections of $\mathscr{E}_1$ and $\mathscr{E}_2$, respectively. Suppose that $L_1$ and $L_2$ are elliptic. An elliptic operator on a compact manifold has a finite number of zero modes. One can show that the zero modes of $D_+$ and $D_-$ coincide with zero modes of $L_1$ and $L_2$, respectively, see Exercise 5.7. Therefore, the numbers of zero modes of $D_+$ and $D_-$ are finite and we can define the index of the operator $D_+$ as

$$\text{index}(D_+) = N_1 - N_2, \tag{5.78}$$

where $N_1$ is the number of zero modes of $D_+$ and $N_2$ is the number of zero modes of its adjoint.

By using the intertwining relations

$$D_+ L_1 = L_2 D_+, \qquad D_- L_2 = L_1 D_- \tag{5.79}$$

one can show with the help of Exercise 5.8 that all non-zero eigenvalues of $L_1$ and $L_2$ coincide. Since both $L_1$ and $L_2$ are elliptic one can define corresponding heat kernels and calculate their difference

$$K(t, L_1) - K(t, L_2) = \sum_{\lambda_1} e^{-t\lambda_1} - \sum_{\lambda_2} e^{-t\lambda_2} = N_1 - N_2 = \text{index}(D_+), \tag{5.80}$$

where $\lambda_k$ are eigenvalues of $L_k$. Suppose that $L_k$ are Laplace type operators which allow asymptotic expansion (4.9). Then one can expand both sides of (5.80) in power series in $t$ to get

$$\begin{aligned} a_k(D_+) - a_k(D_-) &= 0 \quad \text{for } k \neq n, \\ a_n(D_+) - a_n(D_-) &= \text{index}(D_+). \end{aligned} \tag{5.81}$$

Equation (5.81) is called the index theorem. It shows that certain combinations of the heat coefficients is an integer number and, thus, are topological, or homotopy, invariants. This follows from the properties of the heat kernel coefficients, see (4.56)–(4.58) and (4.70)–(4.75), which depend smoothly on the potential, curvature, extrinsic curvature and other fields which characterize the fiber bundle and the operator. Smooth variations of these quantities cannot change the index. Of course, no "essential" changes are allowed. For example, one cannot replace Dirichlet boundary conditions by the Neumann ones.

Let us discuss the index for a number of examples. Our first example was already presented in Sect. 4.5. The operators $D_\pm$ have been given there by Eq. (4.77), while (5.81) is equivalent to (4.85) and (4.86). The corresponding index is invariant with respect to smooth variations of the potential $V$.

The example above can be modified by choosing $D_+ = e^{\rho(x)}\partial_x$ where $\rho(x)$ is a smooth function. The corresponding adjoint operator is $D_- = -e^{\rho(x)}(\partial_x + \rho'(x))$. As in the previous example we choose Dirichlet boundary conditions on the sections of $\mathscr{E}_1$, i.e. $\varphi_1|_{\partial\mathscr{M}} = 0$. Since $D_-$ maps smooth sections of $\mathscr{E}_2$ to smooth sections of $\mathscr{E}_1$ it means that $D_-\varphi_2$ satisfies the Dirichlet conditions, which is equivalent to a Robin boundary condition for $\varphi_2$, i.e. $(\partial_x + \rho'(x))\varphi_2 = 0$ on the boundary. Both $L_1$ and $L_2$ are of Laplace type and one can use (4.71) and (4.74) to calculate the index by using (4.76) to find the constants $\beta_1^D$ and $\beta_1^N$. One still has

$$\text{index}(D_+) = -1 \tag{5.82}$$

independently of $\rho$. This example shows that the index is invariant under smooth variations of the function $\rho$ which mimics the metric of a base manifold. Here one has a one-dimensional analog of a topological invariance.

Note that there are no zero modes for Dirichlet boundary conditions since $D_+\varphi_1^{(0)} = 0$ yields constant mode $\varphi_1^{(0)}$, and Dirichlet boundary conditions require $\varphi_1^{(0)} = 0$. There is always one zero mode in the Neumann sector, since $D_-\varphi_2^{(0)} = 0$ always has a solution $\varphi_2^{(0)} = e^{-\rho}$ which automatically satisfies the Robin conditions given above.

The last example is related to the Dirac operator $\not{D}$. Consider a spin bundle over an even-dimensional manifold without boundaries, see Sect. 1.5. A generic Dirac operator is locally defined by (3.6). Let us assume that the zeroth order part $V$ is such that

$$\gamma_* \not{D} = -\not{D}\gamma_*, \tag{5.83}$$

where $\gamma_*$ is the chirality matrix, see (1.59). One can always choose a basis such that

$$\gamma_* = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{5.84}$$

where $I$ is a unit matrix. Then, due to (5.83), the Dirac operator has the form

$$\not{D} = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \tag{5.85}$$

with some operators $D_+$ and $D_-$ called chiral operators. It is assumed that the Dirac operator is self-adjoint. This implies that $D_+ = D_-^\dagger$. The bundle $\mathscr{E}_1$ corresponds to the spinors of positive chirality (positive eigenvalues of $\gamma_*$), and $\mathscr{E}_2$ corresponds to the negative chirality spinors. The projectors on positive and negative chirality spinors are

$$P_\pm = \frac{1}{2}(1 \pm \gamma_*). \tag{5.86}$$

The index measures the difference of numbers of positive and negative chirality zero modes of the Dirac operator and takes the following form:

$$\text{index}(D_+) = \mathrm{Tr}\big(e^{-tD_-D_+}\big) - \mathrm{Tr}\big(e^{-tD_+D_-}\big) = \mathrm{Tr}\big(\gamma_* e^{-t\not{D}^2}\big). \tag{5.87}$$

Formula (5.87) has important physical applications related to the so-called chiral anomalies. Explicit calculations are given in Sect. 8.2.

## 5.10 Spectral Functions Related to Transformations of Chiral Operators

The chiral operators $D_+$ and $D_-$ which have been discussed in the last example of the previous section are considered in a number of theoretical settings. The definition of the determinants of such operators is not possible because the operators act between sections of different bundles. For example, $D_+$ maps smooth sections of $\mathscr{E}_1$ (positive chirality spinors) to smooth sections of $\mathscr{E}_2$ (negative chirality spinors). Moreover, typical form of the leading part of $D_+$ is $\partial_1 + i\partial_2$ (as for the Dirac operator in two dimensions with an appropriate choice of $\gamma$-matrices). The eigenvalues of such an operator are spread over the whole complex plane, so that it is hardly possible to define any spectral function.

In physical applications it is sometimes enough to define transformation properties of a determinant of $D_+$ without defining determinants of chiral operators themselves. This task is much easier and can be resolved with the help of spectral functions, in analogy to the methods of Sect. 5.7. By anticipating a more physical discussion of the next Part of this book, we may say that this corresponds to dealing directly with the anomalies without calculating the effective action.

To study spectral properties of the chiral operator $D \equiv D_+$ we introduce an auxiliary chiral operator $\bar{D}$ which maps sections of $\mathscr{E}_2$ to sections of $\mathscr{E}_1$. The operator $\bar{D}$ is not specified. The only requirement is that another auxiliary operator $D\bar{D}$ is of Laplace type. The role of $\bar{D}$ is to compensate the main part of the phase of spectrum of $D_+$, so that only relatively small fluctuations of the phase remain. By analogy with (5.85) we also introduce a Dirac type operator

$$\hat{D} = \begin{pmatrix} 0 & \bar{D} \\ D & 0 \end{pmatrix}, \tag{5.88}$$

which acts on smooth sections of the bundle $\mathscr{E}_1 + \mathscr{E}_2$. Consider the spectral function $\ln\det\hat{D}$, where the Ray-Singer formula is used to define the determinant. We shall study transformation properties of $\ln\det\hat{D}$ under variations of $D$ when $\bar{D}$ is assumed to be fixed.

To proceed we specify transformation as

$$\delta_\pm D = \mathscr{G} D \pm D\mathscr{G}, \tag{5.89}$$

where a generator $\mathscr{G}$ is some operator which maps sections of $\mathscr{E}_1$ to sections of $\mathscr{E}_1$ and sections of $\mathscr{E}_2$ to sections of $\mathscr{E}_2$. The operator $\mathscr{G}$ should satisfy the same restrictions as $\mathscr{O}$ in Sect. 5.7. After simple algebra one gets

$$\delta_\pm \operatorname{Tr} e^{-tD\bar{D}} = -t \operatorname{Tr} \mathscr{G}\big(D\bar{D}e^{-tD\bar{D}} \pm \bar{D}De^{-t\bar{D}D}\big), \tag{5.90}$$

where we used the Duhamel formula (5.62). In terms of the operator $L \equiv \hat{D}^2$, Eq. (5.90) takes the form similar to (5.63),

$$\begin{aligned}\delta_\pm \operatorname{Tr} e^{-tL} &= \delta_\pm \operatorname{Tr} e^{-tD\bar{D}} + \delta_\pm \operatorname{Tr} e^{-t\bar{D}D} \\ &= 2\delta_\pm \operatorname{Tr} e^{-tD\bar{D}} = -t \operatorname{Tr}\big(\mathscr{O}_\pm L e^{-tL}\big),\end{aligned} \tag{5.91}$$

$$\mathcal{O}_- = -2\gamma_\star \mathcal{G}, \qquad \mathcal{O}_+ = 2\mathcal{G}, \tag{5.92}$$

with $\gamma_\star$ defined in (5.84). We have noticed here that the heat traces of the operators $D\bar{D}$ and $\bar{D}D$ differ by a constant associated with the number of zero modes. By using now (5.71) and assuming that $\bar{D}$ is fixed one gets the following transformation law:

$$\delta_\pm \ln\det \hat{D} = \frac{1}{2}\delta \ln\det L = \frac{1}{2}\zeta(0, L, \mathcal{O}_\pm). \tag{5.93}$$

It is convenient to consider variations of the absolute value $|\det \hat{D}|$ and of the phase $\Phi(\hat{D})$ of the determinant

$$\ln|\det \hat{D}| = \frac{1}{2}(\ln\det \hat{D} + \ln\det \hat{D}^+) = \frac{1}{2}(\ln\det DD^+ + \ln\det \bar{D}\bar{D}^+), \tag{5.94}$$

$$\Phi(\hat{D}) = \frac{i}{2}(\ln\det \hat{D}^+ - \ln\det \hat{D}). \tag{5.95}$$

When applying (5.93) to (5.94), (5.95) it is convenient to fix $\bar{D}$ when the variation is performed as $\bar{D} = D^+ = D_-$. Then $\hat{D} = \not{D}$ and $L = \not{D}^2$ is a Hermitian operator. This implies that

$$\delta_\pm \ln|\det \hat{D}| = \frac{1}{4}\zeta\big(0, \not{D}^2, (\mathcal{O}_\pm + \mathcal{O}_\pm^+)\big), \tag{5.96}$$

$$\delta_\pm \Phi(\hat{D}) = \frac{i}{4}\zeta\big(0, \not{D}^2, (\mathcal{O}_\pm^+ - \mathcal{O}_\pm)\big). \tag{5.97}$$

It follows then from (5.92) that the absolute value does not transform when the generator is anti-Hermitian, $\mathcal{G}^+ = -\mathcal{G}$, transformation of the phase vanishes for Hermitian generators $\mathcal{G}^+ = \mathcal{G}$. Note that according to (5.94) variations of the absolute value at fixed $\bar{D}$ can be written as

$$\delta_\pm \ln|\det \hat{D}| = \frac{1}{2}\delta_\pm \ln\det DD^+ = \frac{1}{2}\delta_\pm \ln\det \not{D}. \tag{5.98}$$

Thus, one can also get transformation (5.96) for the absolute value by using results of Sect. 5.7, see Exercise 5.10. Equations (5.93), (5.96), (5.97) will be used in Chap. 8.

An obvious drawback of the method described above is that it depends on the choice of the auxiliary operator $\bar{D}$.

## 5.11 Literature Remarks

The Riemann zeta-function was introduced by B. Riemann in 1859. Besides of applications to quantum theory it plays a prominent role in mathematics, in particular in the number theory. Riemann conjectured that all zeros of $\zeta(s)$ (in addition to its zeros at negative integers) lie on the line $\Re s = 1/2$ and that the distribution of these zeros is related to the distribution of prime numbers. The proof of the Riemann hypothesis is included in Hilbert's list of problems and has not been given so far.

There are a number of monographs devoted to different aspects of the zeta-function of differential operators and their physical applications. Among them are the monographs by Elizalde et al. [100, 101]. A generalization of formula (5.28) for the zeta-function $\zeta(s; L, Q)$, see (5.33), can be found in [134].

The references for the Weyl formula (5.39) are standard textbooks, see e.g. [237]. The smoothing of the counting function $N(\lambda)$ and the spectral density $\rho(\lambda)$ is discussed e.g. in [22, 117]. The smoothing in terms of the Riesz means (5.40) was used by Fulling [118], while the form of the sub-leading terms in (5.43) as distributions has been pointed out by Dowker [91].

The multiplicative anomaly was introduced by Kontsevich and Vishik [174]. For more references and physical applications, see e.g. [102, 139].

The index theorem is perhaps one of the most beautiful and powerful discoveries of mathematics in 20th century. In 2004 M. Atiyah and I.M. Singer, the authors of the theorem, have been awarded the Abel Prize "for their discovery and proof of the index theorem, bringing together topology, geometry and analysis, and their outstanding role in building new bridges between mathematics and theoretical physics".

The topological invariants which appear in the index theory are very useful in a number of physical applications. For instance, for analyzing classical solutions of field equations since the invariants determine so-called topological charges of the solutions (the monopole magnetic charge, e.g.). They are also important in studying quantum corrections to such solutions, especially in the case of supersymmetric models, see Sect. 9.6. The presence of a topological charge indicates in many cases that some global symmetries of classical theory are broken at the quantum level. This effect is a particular manifestation of a quite general phenomenon, quantum anomalies which are discussed in Chap. 8.

The auxiliary Dirac operator $\hat{D}$ introduced in Sect. 5.10 in relation to transformations of chiral operators was suggested by Alvarez-Gaumé and Ginsparg in [6, 7] for operators in external gauge fields. More references on chiral operators are given in Sect. 8.8.

Recommended Exercises are 5.5 and 5.6.

## 5.12 Exercises

**Exercise 5.1** Let $L$ be a Laplace operator $\Delta$, see Eq. (3.4), on a unit $n$-sphere $S^n$. Show that the series $\sum_\lambda \lambda^{-s}$, where $\lambda$ are eigenvalues of $\Delta$, converges if $\Re s > n/2$.

**Exercise 5.2** By using the spectrum of the Laplace operator $\Delta$ on a unit sphere $S^2$, the zeta-function $\zeta(s, \Delta + 1/4)$, and relation (4.15) verify the first terms in asymptotic expansion (4.15).

**Exercise 5.3** Consider a (1 + 1)-dimensional theory of a complex massless scalar field on a circle with "twisted" periodicity condition $\varphi(t, x + l) = e^{ib}\varphi(t, x)$, where $b$ is a real constant, $0 < b < 2\pi$. Find the vacuum energy in this model by using the $\zeta$-function regularization.

**Exercise 5.4** By using results of Exercise 3.3 find the heat coefficient $a_2(\Delta)$ for the heat kernel of the vector Laplacian (3.5) on the unit two-sphere $S^2_\beta$ with conical singularities, see Eq. (1.97). Use this result to find the same coefficient on a cone.

**Exercise 5.5** Consider a quantum theory of a charged scalar field (1.68) interacting with a classical constant magnetic field. Field equation (1.69) can be written as

$$(D^\mu(A)D^\mu(A) - m^2)\varphi = (-\partial_t^2 - L(A))\varphi = 0, \tag{5.99}$$

$$L(A) = -(\partial_k + ieA_k)(\partial^k + ieA^k) + m^2, \tag{5.100}$$

where $k = 1, \dots, n$. The spatial part of the wave operator, $L(A)$, is a Laplacian.

Express the zeta-function $\zeta(s; L(A))$ in dimensions $n = 2$ and $n = 3$ in terms of the generalized Riemann zeta-function (5.4). Suppose for simplicity that the scalar field is located in a region with a finite but large volume $V$ (neglect boundary effects). The strength of the magnetic field is $B$.

**Exercise 5.6** Consider a one-parameter family of second order elliptic operators $L(\alpha)$ defined in Sect. 5.7. Prove the following formula for the heat coefficients of this family:

$$\frac{d}{d\alpha} a_p(L_\alpha) = \frac{p-n}{2} a_p(L_\alpha, \mathscr{O}), \tag{5.101}$$

where $n$ is the dimensionality of the corresponding base manifold.

**Exercise 5.7** Prove that zero modes of $D_+$ coincide with zero eigenmodes of $L_1$, see Sect. 5.9.

**Exercise 5.8** Use intertwining relations (5.79) to demonstrate that the non-zero eigenvalues of $L_1$ and $L_2$ coincide.

**Exercise 5.9** Find variation of the phase of the determinant of the chiral part of the Dirac operator $\not{D}(B) = i\gamma^\mu(\partial_\mu + B_\mu)$ on a flat even-dimensional manifold with $SU(N)$ gauge field $B_\mu$. Take transformation of the field in the following form:

$$B'_\mu(x) = U^\dagger(x)(B_\mu(x) + \partial_\mu)U(x), \tag{5.102}$$

where $U$ belongs to the $SU(N)$ group. Carry out explicit computations for dimensions of the base manifold $n = 2$ and $n = 4$.

**Exercise 5.10** Use formula (5.71) to find transformation of $\ln\det\not{D}$, where $\not{D}$ is given by (5.85), and the chiral part of the operator transforms as

$$\delta_\pm D_+ = \mathscr{G} D_+ \pm D_+ \mathscr{G}, \tag{5.103}$$

see (5.89). Check that the result coincides with transformation (5.96) which follows from relation (5.98) between $\det \not{D}$ and the determinant of the auxiliary operator $\hat{D}$.

# Chapter 6
# Non-linear Spectral Problems

## 6.1 Formulation of the Problem

As it was discussed in Sect. 2.5, wave equations on stationary backgrounds allow separation of variables and take the form of spectral problem (2.43) for single-particle energies. A straightforward generalization of (2.43) is

$$\big(P_0\omega^k + P_1\omega^{k-1} + \cdots + P_{k-1}\omega + P_k\big) f_\omega(x) = 0, \tag{6.1}$$

where $P_l$ is a partial differential operator of the $l$-th order and operators $P_l$, $P_m$ may not commute for $l \neq m$. Problems like (6.1) are non-linear with respect to the spectral parameter $\omega$. In the mathematical literature they are called polynomial operator pencils [182]. In the present book we call (6.1) non-linear spectral problem (NLSP) of the polynomial type.

The simplest example of NLSP appears in quantum theory in a static classical electric potential. For example, substitution $\varphi(t,x) = e^{-i\omega t}\varphi_\omega(x)$ in the Klein-Gordon equation (1.69) for a charged field in a static gauge potential, $A_\mu\, dx^\mu = \Phi\, dx^0$, yields the problem

$$\big((\omega + e\Phi)^2 + \partial_i\partial^i - m^2\big)\varphi_\omega(x) = 0. \tag{6.2}$$

Equation (6.2) can be brought to form (6.1), where $P_2 = \partial_i\partial^i - m^2 + e^2\Phi^2$, $P_1 = 2e\Phi$, $P_0 = 1$. The operators $P_1$ and $P_2$ do not commute if the gauge potential $\Phi$ depends on coordinates.

A less trivial example of (6.1) in quantum field theory will be considered latter. The non-linear spectral problems are quite common. They also appear in quantum mechanics when the potential in the Hamilton operator depends on the energy.

The aim of the present Chapter is to show how the methods of the spectral geometry are extended to NLSP's. We introduce a spectral function analogous to the heat trace, and show that its asymptotics are expressed in terms of integrals of local geometrical invariants. Sometimes, the spectral problem may have a polynomial structure like (6.1) only at large values of the spectral parameter. We call these problems *asymptotically polynomial* NLSP and show how to find asymptotics in this case. An example of asymptotically polynomial NLSP is provided by noncommutative theories and it is discussed in Chap. 11.

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## 6.2 A Method of Finding the Spectrum

There is a method to find the spectrum of an NLSP by reducing it to linear spectral problems. We focus on a second order problem like (2.43) and bring it to a slightly different form

$$\left[\omega^2 - L(\omega)\right]\varphi_\omega(x) = 0, \tag{6.3}$$

$$L(\omega) = L_2 + \omega L_1 + \omega^2 L_0. \tag{6.4}$$

It is assumed that $L(\omega)$ acts on the space $\mathrm{L}^2$ associated to a vector bundle over a compact $n$-dimensional Riemannian manifold $\mathscr{M}$ with metric $h_{ik}$. We take $L(\omega)$ in the following form relevant for applications:

$$L(\omega) = -(\nabla_k + iA_k + i\omega a_k)(\nabla^k + iA^k + i\omega a^k) + \omega B + V. \tag{6.5}$$

Here the index $k$ is raised and lowered with the help of metric $h_{ik}$, $\nabla_k$ is a connection on $\mathscr{M}$, a $A_k$, $a_k$, $B$ and $V$ are some matrix-valued functions. The base manifold $\mathscr{M}$ has the meaning of a constant-time section of a physical spacetime. The inner product in $\mathrm{L}^2$ is defined as

$$(f_1, f_2) = \int_{\mathscr{M}} \sqrt{h}\, d^n x\, f_1^* \eta f_2, \tag{6.6}$$

where $\eta$ is a Hermitian matrix. We suppose that $L(\omega)$ is a self-adjoint operator for real values of the parameter $\omega$. In applications, $\eta$ may not be positive-definite. For example, for vector fields $\eta$ is the Minkowski metric. Therefore, the inner product (6.6) is Hermitian, but may be indefinite, i.e. the space $\mathrm{L}_2$ may have an indefinite metric.

Consider now the spectral problem associated to (6.3)

$$L(\omega)\varphi^{(\omega)}_{\lambda_k} = \lambda_k(\omega)\varphi^{(\omega)}_{\lambda_k}, \tag{6.7}$$

where $\omega$ is a real parameter and $k$ enumerates the eigenvalues. The eigenvalues $\lambda_k(\omega)$ are real because $L(\omega)$ is Hermitian. Moreover, if $L(\omega)$ is a positive elliptic operator, one can show that its spectrum is bounded from below.

Given (6.7), the approach to (6.3) is simple: one should find $\lambda_k(\omega)$ for any $\omega$ and then look for the roots of the algebraic equation

$$\chi(\omega, \lambda_k) = 0, \tag{6.8}$$

$$\chi(\omega, \lambda_k) = \omega^2 - \lambda_k(\omega). \tag{6.9}$$

For further purposes we introduce the following function:

$$\chi'(\omega_k) = \partial_\omega \chi(\omega_k, \lambda_k), \quad \text{where } \omega_k^2 = \lambda_k(\omega_k). \tag{6.10}$$

It is assumed that for a fixed branch of eigenvalues $\lambda_k(\omega)$ the derivative over $\omega$ is taken and after that the result is considered at one of the roots of (6.8). In what follows we denote the eigenvalues of NLSP by $\omega$ instead of $\omega_k$. Different eigenvalues will be denoted by different letters, say $\omega$ and $\sigma$.

The role of function $\chi'$ becomes evident if we consider the wave equation

$$\left[(1 - L_0)\partial_t^2 + L_2 + iL_1\partial_t\right]\varphi(t, x) = 0. \tag{6.11}$$

As was explained in Sect. 2.1, normalization of modes in a relativistic theory is determined by the relativistic product rather than inner product (6.6). The corresponding product for (6.11) is

$$\langle\varphi, \psi\rangle = i(\varphi, \dot{\psi}) - i(\dot{\varphi}, \psi) - (\varphi, L_1\psi) - i(\varphi, L_0\dot{\psi}) + i(\dot{\varphi}, L_0\psi), \tag{6.12}$$

where $\dot{\varphi} = \partial_0\varphi$. This bilinear form does not depend on time on the solutions to (6.11). According to (6.12), the product of any two solutions, $\varphi_\omega(t, x) = e^{-i\omega t}\varphi_\omega(x)$, $\varphi_\sigma(t, x) = e^{-i\sigma t}\varphi_\sigma(x)$ (where $\varphi_\omega(x)$ and $\varphi_\sigma(x)$ are eigenfunctions to (6.3)), can be written in the form

$$\langle\varphi_\omega, \psi_\sigma\rangle = (\omega + \sigma)(\varphi_\omega, (1 - L_0)\psi_\sigma) - (\varphi_\omega, L_1\psi_\sigma). \tag{6.13}$$

An important relation follows from (6.13),

$$\langle\varphi_\omega, \psi_\sigma\rangle = \delta_{\omega\sigma}\chi'(\omega)(\varphi_\omega, \psi_\omega), \tag{6.14}$$

where $\delta_{\omega\sigma} = 0$, if $\omega \neq \sigma$, and $\delta_{\omega\sigma} = 1$, if $\omega = \sigma$. The derivation of (6.14) is left as Exercise 6.1.

The quantity $\chi'(\omega)$ relates the two norms: the norm associated to the inner product $(,)$ in $L^2$ and the norm introduced with the help of the product $\langle,\rangle$. It follows that two eigenfunctions of (6.3) are orthogonal with respect to $\langle,\rangle$ if they are orthogonal with respect to $(,)$.

The analysis of spectral asymptotics of NLSP can be simplified if we require that

$$\chi'(\omega) = \varepsilon(\omega)|\chi'(\omega)|, \tag{6.15}$$

where $\varepsilon(\omega)$ is the sign function.

The requirement (6.15) is related to physical features. If (6.6) is positive definite, say, $\eta = 1$, it follows from (6.15), (6.14) that the relativistic norm $\langle\varphi_\omega, \varphi_\omega\rangle$ is positive (negative) for modes with positive (negative) frequency. This property guarantees that all solutions have positive canonical energy $H[\varphi]$, see Sect. 2.5 and Eqs. (2.44), (2.45).

Note, that if the field equations are invariant under the charge conjugation, the eigenvalues $\lambda(\omega)$ are symmetric functions of $\omega$.

The problems like (6.3) may have complex eigenvalues $\omega$. The eigenfunctions $\varphi_\omega$ with complex frequencies have zero norm (6.12), $\langle\varphi_\omega, \varphi_\omega\rangle = 0$. It means that complex frequency modes do not contribute to the energy ($H[\varphi_\omega] = 0$) and they should not be quantized.

## 6.3 Spectral Geometry of Non-linear Spectral Problems

Let $\omega$ be real eigenvalues of (6.3). Consider the spectral function (a 'pseudo-trace'):

$$K(t) = \frac{1}{2}\sum_\omega e^{-t\omega^2}, \quad t > 0, \tag{6.16}$$

where the sum is taken over all eigenvalues including negative $\omega$. The coefficient $1/2$ in the right hand side of (6.3) is introduced for the convenience: if $L(\omega) = L_2$ the pseudo-trace $K(t)$ coincides with the trace of the heat kernel of $L_2$.

It turns out that the asymptotic expansion of the pseudo-trace has the form of the standard heat kernel asymptotic (4.9),

$$K(t) \sim \sum_{p=0}^{\infty} a_p t^{\frac{p-n}{2}}. \tag{6.17}$$

The coefficients $a_p$ are analogs of the heat coefficients, and they are related to the heat coefficients $a_p(\omega)$ for the operator $L(\omega)$. If $\omega$ is fixed, one can write

$$K(L(\omega); t) = \mathrm{Tr}\, e^{-tL(\omega)} \sim \sum_{p=0}^{\infty} a_p(\omega) t^{\frac{p-n}{2}}. \tag{6.18}$$

The fact that the operator $L(\omega)$ has a polynomial dependence on $\omega$, see (6.4), implies a similar structure for $a_p(\omega)$,

$$a_{2k}(\omega) = \sum_{m=0}^{k} a_{m,k}\omega^m, \quad a_{2k+1}(\omega) = \sum_{m=0}^{k} b_{m,k}\omega^m. \tag{6.19}$$

The highest power of $\omega$ in $a_p(\omega)$ is determined by the presence of the term $\omega B$ in $L(\omega)$. We show below that the coefficients for the pseudo-trace can be represented as

$$a_{2k} = \sum_{m=k}^{2k} (-1)^{k-m} \frac{\Gamma(-\frac{n}{2} + m)}{\Gamma(-\frac{n}{2} + k)} a_{2(m-k),m}, \tag{6.20}$$

$$a_{2k+1} = \sum_{m=k}^{2k} (-1)^{k-m} \frac{\Gamma(-\frac{n-1}{2} + m)}{\Gamma(-\frac{n-1}{2} + k)} b_{2(m-k),m}. \tag{6.21}$$

Therefore, the asymptotic expansion of (6.17) is given in terms of integrals of local geometrical invariants of background fields and the notion of the spectral geometry is fully applicable to the polynomial NLSP's.

## 6.4 Derivation of Asymptotic Expansions

Suppose that the operator $L(\omega)$ is a Laplace-type operator for real $\omega$ and the spectrum of $L_2 = L(0)$ is strictly positive. The proof of (6.17) is based on a relation between $K(t)$ and $K(L(\omega); t)$. To find this relation one can represent (6.16) as follows:

$$K(t) = \frac{1}{2} \int_{-\infty}^{\infty} d\omega \sum_{\lambda_k(\omega)} \delta(\chi(\omega, \lambda_k))\, |\partial_\omega \chi(\omega, \lambda_k)| e^{-t\omega^2}. \tag{6.22}$$

By using (6.15) and the integral representation for the $\delta$-function the right hand side of (6.22) can be brought to the form

$$K(t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\omega\,\varepsilon(\omega) \sum_{\lambda_k(\omega)} \int_{-\infty}^{\infty} dx\, e^{ix\chi(\omega,\lambda_k)} \partial_\omega \chi(\omega,\lambda_k) e^{-t\omega^2}$$
$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} d\omega\,\varepsilon(\omega) \int_C dz\, e^{-\omega^2(t-iz)} \left(2\omega + \frac{1}{iz}\partial_\omega\right) K(L(\omega); iz). \tag{6.23}$$

The integration contour over $x$ in (6.23) was deformed in a contour $C$ lying in the complex plane and going from $-i\epsilon - \infty$ to $-i\epsilon + \infty$ where $\epsilon$ is a small positive parameter. To proceed, we represent $K(L(\omega); t)$ in an integral form

$$K(L(\omega); t) = \int_\mu^\infty e^{-t\lambda} \rho(\lambda,\omega)\, d\lambda, \tag{6.24}$$

where $\rho(\lambda,\omega)$ is the spectral density, see Sect. 5.4. Parameter $\mu$ is chosen to be smaller than the lowest eigenvalue $\lambda_0(\omega)$. One can use now (6.24) in (6.23) to get

$$K(t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\omega\,\varepsilon(\omega) \int_C dz\, e^{-\omega^2(t-iz)} \int_\mu^\infty d\lambda\, e^{-iz\lambda}$$
$$\times \left(2\omega\rho(\lambda,\omega) + \frac{1}{iz}\partial_\omega\partial_\lambda N(\lambda,\omega)\right), \tag{6.25}$$

where $N(\lambda,\omega)$ is the counting function

$$N(\lambda,\omega) = \int_\mu^\lambda d\sigma\, \rho(\sigma,\omega). \tag{6.26}$$

The last term in the brackets in (6.25) can be integrated by parts over $\lambda$. Then the integral over $z$ results in the delta-function $\delta(\lambda - \omega^2)$, and one gets the final expression

$$K(t) = \int_0^\infty d\lambda\, e^{-\lambda t} \rho(\lambda), \tag{6.27}$$

$$\rho(\lambda) = \frac{1}{2}\left(\tilde{\rho}(\lambda,\sqrt{\lambda}) + \tilde{\rho}(\lambda,-\sqrt{\lambda})\right), \tag{6.28}$$

$$\tilde{\rho}(\lambda,\omega) = \rho(\lambda,\omega) + \frac{1}{2\omega}\partial_\omega N(\lambda,\omega). \tag{6.29}$$

As was explained in Sect. 5.4, the asymptotic expansion of the heat trace can be related to the asymptotic properties of the smoothed spectral function (see Eq. (5.40)). To establish this relation, we define the function

$$\rho_\alpha(\lambda,\omega) = \frac{1}{\Gamma(\alpha)}\int_\mu^\lambda (\lambda-\sigma)^{\alpha-1} \rho(\sigma,\omega)\, d\sigma, \tag{6.30}$$

where $\alpha$ is a regularization parameter, $\alpha \neq k/2$, where $k$ is an integer. At large $\lambda$, the spectral function behaves according to (5.42),

$$\rho_\alpha(\lambda,\omega) \sim \sum_{p=0}^{\infty} a_p(\omega) \frac{\lambda^{(n-p)/2+\alpha-1}}{\Gamma(\frac{n-p}{2}+\alpha)}. \tag{6.31}$$

To get asymptotics of the spectral density (6.28), the smoothing procedure is applied to (6.29),

$$\tilde{\rho}_\alpha(\lambda,\omega) = \rho_\alpha(\omega,\lambda) + \frac{1}{2\omega}\partial_\omega \rho_{\alpha+1}(\lambda,\omega). \tag{6.32}$$

It follows from (6.31) that at large $\lambda$

$$\frac{1}{2}(\tilde{\rho}_\alpha(\lambda,\omega) + \tilde{\rho}_\alpha(\lambda,-\omega)) \sim \sum_{p=0}^{\infty} \tilde{a}_p(\omega) \frac{\lambda^{(n-p)/2+\alpha-1}}{\Gamma(\frac{n-p}{2}+\alpha)}, \tag{6.33}$$

$$\tilde{a}_p(\omega) = \frac{1}{2}\left[a_p(\omega) + a_p(-\omega) + \frac{1}{2\omega}\partial_\omega(a_{p+2}(\omega) + a_{p+2}(-\omega))\right], \tag{6.34}$$

where the property $\partial_\omega a_0(\omega) = 0$ has been taken into account. Equation (6.19) implies that

$$\tilde{a}_{2k}(\omega) = \sum_{m=0}^{\infty} \tilde{a}_{2m,k}\omega^{2m}, \quad \tilde{b}_{2k+1}(\omega) = \sum_{m=0}^{\infty} \tilde{b}_{2m,k}\omega^{2m}, \tag{6.35}$$

$$\begin{aligned} \tilde{a}_{2m,k} &= a_{2m,k} + (m+1)a_{2(m+1),k+1}, \\ \tilde{b}_{2m,k} &= b_{2m,k} + (m+1)b_{2(m+1),k+1}, \end{aligned} \tag{6.36}$$

where $a_{2m,k}$, $b_{2m,k}$ are assumed to vanish for $2m > k$. Then

$$\frac{1}{2}\big(\tilde{\rho}_\alpha(\lambda,\sqrt{\lambda}) + \tilde{\rho}_\alpha(\lambda,-\sqrt{\lambda})\big) \sim \sum_{p=0}^{\infty} a_p^{(\alpha)} \frac{\lambda^{(n-p)/2+\alpha-1}}{\Gamma(\frac{n-p}{2}+\alpha)}. \tag{6.37}$$

Coefficients $a_p^{(\alpha)}$ can be found with the help of (6.35), (6.36). After some algebra one gets

$$a_{2k}^{(\alpha)} = \sum_{m=k}^{2k} (-1)^{k-m} \frac{\Gamma(-\frac{n}{2}+m-\alpha)}{\Gamma(-\frac{n}{2}+k-\alpha)} a_{2(m-k),m}, \tag{6.38}$$

$$a_{2k+1}^{(\alpha)} = \sum_{m=k}^{2k} (-1)^{k-m} \frac{\Gamma(-\frac{n-1}{2}+m-\alpha)}{\Gamma(-\frac{n-1}{2}+k-\alpha)} b_{2(m-k),m}. \tag{6.39}$$

Asymptotics of spectral density (6.28) is obtained from (6.37) in the limit $\alpha \to 0$. To this aim one should treat its coefficients as generalized functions, in the way explained in Sect. 5.4,

$$\rho(\lambda) = \frac{1}{2}\lim_{\alpha\to 0}\big(\tilde{\rho}_\alpha(\lambda,\sqrt{\lambda}) + \tilde{\rho}_\alpha(\lambda,-\sqrt{\lambda})\big) \sim \sum_{p=0}^{\infty} a_p\, \partial_\lambda^{\frac{p-n}{2}} \delta(\lambda). \tag{6.40}$$

Here the symbol $\partial_\lambda^\gamma$ for $\gamma \neq n$ denotes the fractional derivative defined by (5.42). The coefficients $a_p \equiv a_p^{(0)}$ are given by relations (6.20), (6.21).

By comparing (6.40) with (5.43), one can conclude that the series exactly corresponds to the short $t$ pseudo-trace asymptotic (6.17). This completes the proof our statements in Sect. 6.3.

## 6.5 An Example

Let us demonstrate how the obtained asymptotics can be used in physical applications. As an example, we consider a conformally coupled scalar field,

$$\left(-\nabla^2 + \frac{1}{6}R\right)\varphi = 0 \tag{6.41}$$

in a rotating frame of reference in the Einstein universe $\mathrm{R}^1 \times S^3$. Here $R = 6/\rho^2$ is the scalar curvature and $\rho$ is the radius of the hyper sphere $S^3$. In what follows we put $\rho = 1$. The corresponding background metric is

$$ds^2 = -(dx^0)^2 + \sin^2\theta\, d\psi_1^2 + \cos^2\theta\, d\psi_2^2 + d\theta^2, \tag{6.42}$$

where $0 \le \theta \le \pi/2$, $0 \le \psi_1, \psi_2 \le 2\pi$. In the frame which rotates along $\psi_1$ with the angular velocity $\Omega$ ($|\Omega| < 1/\rho$) the metric takes the form

$$ds^2 = -B(dx^0 + A_1\, d\psi_1)^2 + \frac{\sin^2\theta}{B}\, d\psi_1^2 + \cos^2\theta\, d\psi_2^2 + d\theta^2, \tag{6.43}$$

$$B = 1 - \Omega^2 \sin^2\theta, \quad A_1 = \Omega \sin^2\theta\ B^{-1}, \tag{6.44}$$

where the coordinate $\psi_1$ is changed to $\psi_1 + \Omega x^0$. It is convenient to use conformal invariance of (6.41) and make a conformal transformation of the metric to the following form

$$ds^2 = -(dx^0 + A_1\, d\psi_1)^2 + dl^2, \tag{6.45}$$

$$dl^2 = \frac{1}{B}\left[\frac{\sin^2\theta}{B}\, d\psi_1^2 + \cos^2\theta\, d\psi_2^2 + d\theta^2\right] \equiv h_{jk}\, dx^j\, dx^k. \tag{6.46}$$

The line element $dl^2$ is the metric on a compact three-dimensional manifold $\mathscr{M}$ without boundaries. After the substitution $\varphi(x^0, x^i) = e^{-i\omega x^0}\varphi_\omega(x^i)$ in Eq. (6.41) taken on background (6.45) one comes to a non-linear spectral problem (6.3) with respect to $\omega$. An explicit form of operator (6.4) can be easily determined,

$$L(\omega) = -(\nabla^k + i\omega a^k)(\nabla_k + i\omega a_k) + \frac{1}{6}\bar{R} + \frac{1}{24}F^{jk}F_{jk}. \tag{6.47}$$

Here $F_{jk} = A_{k,j} - A_{j,k}$ and $A_j\, dx^j = A_1\, d\psi_1$, $\nabla_k$ are the covariant derivatives on $\mathscr{M}$, $\bar{R}$ is the scalar curvature of $\mathscr{M}$.

This physical example is convenient because the spectrum of corresponding NLSP can be found explicitly. If $\omega_n$ are energies of quanta in the non-rotating frame (6.42) the energies in the rotating frame (6.43) are $\omega_{nm} = \omega_n + m\Omega$ where $m$ is the projection of the angular momentum on the rotation axis. For model (6.41) the spectrum is $\omega_n = n+1$, where $n = 0, 1, \ldots,$ which follows from the spectrum of the Laplacian on $S^3$. The number $m$ takes values $-n \le m \le n$ and $\omega_{nm}$ has a degeneracy $d_{nm} = n - |m| + 1$ for given $m$ and $n$, see, e.g. [157]. The spectrum of the NLSP also includes negative energies $\omega_{nm} = -\omega_n + m\Omega$. The positive (negative) energy states have a positive (negative) norm defined with respect to the product (6.12). The latter property is easy to understand if we note that signs of $\omega_{nm}$ and $\omega_n$ are the same and (6.12) coincides with the Klein-Gordon product. Thus, the requirement (6.15) is satisfied.

Since positive and negative parts of the spectrum are symmetric, pseudo-trace (6.16) can be written as

$$K(t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (n - |m| + 1)\, e^{-(n+1+m\Omega)^2 t}. \tag{6.48}$$

Its short $t$ expansion should have the form (6.17) where $n = 3$, and $a_{2k=1} = 0$ because $\mathscr{M}$ has no boundaries. The first coefficients for (6.48) can be found explicitly, see Exercise 6.3,

$$a_0 = \frac{\sqrt{\pi}}{4} \frac{1}{1-\Omega^2}, \quad a_2 = -\frac{\sqrt{\pi}}{12} \frac{\Omega^2}{1-\Omega^2}. \tag{6.49}$$

Expressions (6.49) are in agreement with formula (6.20). Indeed, one can check that $a_0$ is proportional to the volume of $\mathscr{M}_3$, see (6.46). For operator (6.47) definitions (6.19) yield

$$\begin{aligned} a_{0,1} &= -\frac{1}{(4\pi)^{3/2} 24} \int_{\mathscr{M}_3} h^{1/2} d^3x\, F^{jk} F_{jk}, \\ a_{2,2} &= -\frac{1}{(4\pi)^{3/2} 12} \int_{\mathscr{M}_3} h^{1/2} d^3x\, F^{jk} F_{jk}. \end{aligned} \tag{6.50}$$

The coefficient $a_2$, as defined by (6.20), reads

$$a_2 = a_{0,1} + \frac{1}{2} a_{2,2} = -\frac{1}{(4\pi)^{3/2} 12} \int_{\mathscr{M}_3} h^{1/2} d^3x\, F^{jk} F_{jk}. \tag{6.51}$$

It coincides exactly with (6.49).

## 6.6 Asymptotically Polynomial Spectral Problems

The short $t$ expansions of the pseudo-trace are determined by the distribution of large eigenvalues. This observation suggests that formulae (6.17), (6.19)–(6.21) can be extended to the case when a non-linear spectral problem has a polynomial form only at asymptotically large values of the spectral parameter.

Consider a generalization of the operator $L(\omega)$, see Eq. (6.5), of the following form

$$L(\omega) = -(\nabla_k + iA_k(\omega))(\nabla^k + iA^k(\omega)) + V(\omega), \tag{6.52}$$

where the gauge connections $A_k(\omega)$ and the potential term $V(\omega)$ are some sufficiently smooth functions of the spectral parameter which admit the following asymptotics:

$$A_k(\omega) \sim A_k + \omega a_k + \text{e.s.t.}, \tag{6.53}$$

$$V(\omega) \sim V + \omega B + \text{e.s.t.} \tag{6.54}$$

valid at large $|\omega|$ up to exponentially small terms ("e.s.t."). The operator $L(\omega)$ in this limit takes form (6.4)

$$L(\omega) \sim L_2 + \omega L_1 + \omega^2 L_0 + \text{e.s.t.} \tag{6.55}$$

We call spectral problems which reduce to (6.3) up to the exponentially small terms asymptotically polynomial NLSP. Relations (6.53)–(6.54) imply that at large $|\omega|$ the heat coefficients of operator (6.52) have a structure similar to (6.19), i.e.

$$a_{2k}(\omega) = \sum_{m=0}^{k} a_{m,k}\omega^m + \text{e.s.t.}, \quad a_{2k+1}(\omega) = \sum_{m=0}^{k} b_{m,k}\omega^m + \text{e.s.t.} \tag{6.56}$$

In this case short $t$ expansion (6.17) of the pseudo-trace must hold and, moreover, the coefficients $a_p$ are still expressed in the form (6.20), (6.21) with the help of quantities $a_{m,k}$ and $b_{m,k}$ from (6.56).

Examples of asymptotically polynomial NLSP which satisfy these properties appear in noncommutative theories, see Chap. 11. One of such examples will be described in due course.

## 6.7 Literature Remarks

The spectral theory of polynomial operator pencils [182] is a field of mathematics where important pioneering results were established by Keldysh [168]. In addition to applications we study here, quadratic and more general operator polynomials appear in other physical problems, for instance, in oscillations of a viscous fluid, Schrödinger equation with energy-dependent potential and etc.

Asymptotic expansions (6.17), (6.20), (6.21) were derived in [123]. A number of consequences of these relations are considered in Exercises 7.8, 7.9 to Chap. 7. This Chapter also contains some physical applications of NLSP such as high-temperature asymptotics in quantum field models, see Exercise 7.19.

## 6.8 Exercises

**Exercise 6.1** Prove the relation between the two products

$$\langle \phi_\omega, \psi_\sigma \rangle = \delta_{\omega\sigma} \chi'(\omega)(\phi_\omega, \psi_\omega)$$

for two eigenfunctions of NLSP, see the discussion in Sect. 6.2.

**Exercise 6.2** Consider the problem

$$[\omega^2 - L_2]\varphi_\omega = 0, \tag{6.57}$$

where $L_2$ is a second order positive elliptic operator on a compact space. We suppose that the lowest eigenvalue $\lambda_0$ of $L_2$ is positive, $\lambda_0 > 0$. Now, if $\omega$ is replaced to $\omega - \varrho$, where $\varrho$ is a real parameter, the eigenvalue problem becomes non-linear

$$\left[\omega^2 - L(\omega)\right]\varphi'_\omega = 0, \tag{6.58}$$

$$L(\omega) = L_2 - \varrho^2 + 2\varrho\omega, \tag{6.59}$$

where $\varphi'_\omega = \varphi_{\omega+\varrho}$. By assuming that $\varrho^2 < \lambda_0$ check validity of formulas (6.20), (6.21).

**Exercise 6.3** Introduce the zeta-function

$$\zeta(\nu) = \sum_{nm} d_{nm} \omega_{nm}^{-2\nu} \tag{6.60}$$

for the spectrum of the field in the rotating Einstein universe considered in Sect. 6.5. Calculate the coefficients in the pseudo-trace expansion from the zeta-function,

$$a_0 = \frac{\sqrt{\pi}}{2} \lim_{\nu \to 3/2} (\nu - 3/2)\zeta(\nu), \tag{6.61}$$

$$a_2 = \sqrt{\pi} \lim_{\nu \to 1/2} (\nu - 1/2)\zeta(\nu), \tag{6.62}$$

and prove formulas (6.49).

# Part III
# Applications

# Chapter 7
# Effective Action

## 7.1 A Route from Classical to Effective Action

In this Chapter we give an elementary introduction to the method of effective action, explain which applications it is used for and show how the effective action can be calculated in terms of the spectral functions.

It is known that the variational derivatives of a classical action with respect to background fields yield physical quantities $\mathscr{O}(x)$ such as, for example, the stress-energy tensor or the gauge current, see Eqs. (1.22) and (1.71). The effective action, in some sense, is a quantum analog of the classical action whose variations with respect to background fields produce averages of corresponding operators $\langle\mathscr{O}(x)\rangle$. Computations of averages were discussed in Sect. 2.7 in terms of the Green's functions in the framework of the point splitting procedure. Now, we relate this procedure to the effective action.

Suppose that $I[\varphi,\phi]$ is a classical action of classical background fields $\phi$ and some dynamical variables $\varphi$, say, scalar fields. The base manifold is assumed to be Riemannian. The variables $\varphi$ are assumed to be non-interacting (free) fields. Since $I[\varphi,\phi]$ is quadratic in $\varphi$ it can be written as

$$I[\varphi,\phi]=(\varphi,P_E[\phi]\varphi), \tag{7.1}$$

where $P_E[\phi]$ is a Laplace type operator discussed in Sect. 3.1. The notations and relation of $P_E[\phi]$ to classical equations of motions in Lorentzian spacetimes are explained in the next sections.

Let us denote the effective action as $W[\phi]$ and define it by the familiar spectral function

$$W[\phi]=\frac{1}{2}\ln\det P_E[\phi], \tag{7.2}$$

where the quantity $\ln\det P_E[\phi]$ is determined by using some regularization prescription, say, by Ray-Singer formula (5.46). This definition, at least formally, allows one to express the action in terms of a Gauss-type functional integral,

$$e^{-W[\phi]}=\int[D\varphi]e^{-I[\varphi,\phi]}. \tag{7.3}$$

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The integral in the right hand side would be well-defined if $P_E[\phi]$ had a finite number of eigenvalues. In this case it is possible to give a prescription for the integration measure $[D\phi]$, see Exercise 7.1.

In a field theory one is dealing with an infinite number of degrees of freedom. Therefore, the integral in (7.3) is just a symbol which we use for illustrative purposes only. For example, one can introduce with its help the Green's function $G(x,x')$ of the operator $P_E[\phi]$. If $P_E[\phi]$ does not have zero modes the Green's function is defined as $G(x,x') = P_E^{-1}(x,x')$ and can be formally written as

$$G(x,x') = e^{W[\phi]} \int [D\varphi]\varphi(x)\varphi(x')e^{-I[\varphi,\phi]}. \tag{7.4}$$

Which quantum state the correlator $G(x,x')$ corresponds to will be explained latter.

Consider an expectation value of some local operator $\mathscr{O}$. Its calculation can be done by using the point-splitting method discussed in Sect. 2.7. Suppose that $\mathscr{O}$ is defined as a limit (2.75) with some bilinear differential operator $D(x,x')$. Then it follows from (2.76) and (7.4) that

$$\langle \mathscr{O}(x) \rangle = \lim_{x \to x'} D(x,x')G(x,x') = e^{W[\phi]} \int [D\varphi]\mathscr{O}[\varphi,\phi](x)e^{-I[\varphi,\phi]}. \tag{7.5}$$

The quantity $\mathscr{O}[\varphi,\phi](x)$ in (7.5) is just the classical expression for $\mathscr{O}$. If the classical quantity $\mathscr{O}$ is defined by using variational procedure,

$$\mathscr{O}[\varphi,\phi](x) = \frac{\delta I[\varphi,\phi]}{\delta\phi(x)}, \tag{7.6}$$

it follows immediately from (7.5) that the analogous variation formula holds for the quantum average

$$\langle \mathscr{O}(x) \rangle = \frac{\delta W[\phi]}{\delta\phi(x)}. \tag{7.7}$$

This is the reason why $W[\phi]$ is called the effective action.

One of advantages of the effective action method is that (7.7) yields mathematically meaningful expressions because $W[\phi]$ is well-defined by virtue of the Ray-Singer formula. The effective action represents an alternative to the point-splitting method. If $P_E$ were a finite-dimensional matrix one could write with the help of (7.2) and (7.7)

$$\int d^n x\, \langle \mathscr{O}(x) \rangle \delta\phi(x) = \frac{1}{2}\,\mathrm{Tr}\big[\delta_\phi P_E \cdot P_E^{-1}\big]. \tag{7.8}$$

The right hand side of (7.8) has the same structure as the analogous definition in terms of the point-splitting method, see Eq. (7.5).

We have just presented a very crude idea of the effective action. In theories of interacting quantum fields the effective action has a broader applicability and can be defined as a generating functional for certain class of Green's functions. Since we restrict ourselves to free fields, the motivations presented above will be enough for the subsequent analysis.

## 7.2 Statistical Physics

Our first step is to show that the physical meaning of the effective action for positive-definite operators is related to finite-temperature theories. We begin therefore with a brief introduction in statistical physics.

If the background fields are stationary there exist time-independent quantum states which describe a thermal equilibrium of a system at finite temperatures $T$. The parameters of the system can be determined with the help of the *partition function*

$$Z(\beta) = \sum_n e^{-\beta \mathscr{E}_n}, \tag{7.9}$$

where $\beta = T^{-1}$ and the sum goes over all possible states of the Fock space characterized by the energies $\mathscr{E}_n$. Since the zero-point fluctuations are not related to thermal excitations of the system one does not take into account in (7.9) the vacuum energy $E_0$. Therefore, $\mathscr{E}_n = E_n - E_0$, where $E_n$ is a total energy of the Fock state. The series (7.9) converges for $\beta > 0$. This definition can be also written in another form

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta :H:}, \tag{7.10}$$

where $:H: = H - E_0$ is the so-called normally ordered Hamiltonian. A finite-temperature state is a mixed state. The average value of an operator $\mathscr{O}$ in such a state is

$$\langle \mathscr{O} \rangle_\beta = Z^{-1}(\beta)\, \mathrm{Tr}\big(\mathscr{O} e^{-\beta :H:}\big). \tag{7.11}$$

To proceed, it is convenient to introduce the free energy of the system

$$F(\beta) = -\beta^{-1} \ln Z(\beta). \tag{7.12}$$

From (7.11) one can easily find, for example, that the average of the thermal energy is expressed in terms of the free energy,

$$\mathscr{E}(\beta) = \frac{\partial}{\partial \beta}(\beta F(\beta)). \tag{7.13}$$

Another important object is the entropy of the system $S$ which can be inferred from the relation

$$F(\beta) = \mathscr{E} - TS. \tag{7.14}$$

Together with (7.13) this yields

$$S(\beta) = \beta^2 \frac{\partial}{\partial \beta} F(\beta). \tag{7.15}$$

The microscopic meaning of entropy is discussed in a variety of textbooks on statistical mechanics. We are not going to repeat this material here. The purpose of Exercise 7.3 is to show that the entropy measures the number of microscopical states corresponding to given macroscopic parameters of the system.

Our interest is non-interacting field theories. In this case the free energy can be easily related to the spectrum of single-particle energies $\omega_i$ discussed in Sect. 2.5. If the fields are quantized according with the Bose statistics,

$$F(\beta) = \beta^{-1} \sum_{\omega_i} \ln\bigl(1 - e^{-\beta\omega_i}\bigr). \tag{7.16}$$

In the case of Fermi statistics,

$$F(\beta) = -\beta^{-1} \sum_{\omega_i} \ln\bigl(1 + e^{-\beta\omega_i}\bigr). \tag{7.17}$$

The summation here goes over all single-particle energies $\omega_i^{(+)}$ and $\omega_i^{(-)}$. For real fields, $\omega_i^{(+)}$ and $\omega_i^{(-)}$ coincide, and one uses one type of the energies only.

The derivation of (7.16), (7.17) is simple. One has to note that the system of free fields is just a set of harmonic oscillators with frequencies $\omega_i$, see the form of the energy operators, Eqs. (2.46), (2.51). The partition function for a Bose oscillator with a frequency $\omega_i$ is

$$Z_i(\beta) = \sum_{n=0}^{\infty} e^{-\beta\omega_i n} = \frac{1}{1 - e^{-\beta\omega_i}}. \tag{7.18}$$

For Fermi statistics, because of the Pauli principle, the sum terminates,

$$Z_i(\beta) = \sum_{n=0}^{1} e^{-\beta\omega_i n} = 1 + e^{\beta\omega_i}. \tag{7.19}$$

The entire free energy is determined by the partition function (7.10) which is the product of $Z_i(\beta)$.

One may say that a finite-temperature theory is a theory with evolution in an imaginary time. Indeed, the operator generating the time evolution is defined as

$$\hat{U}(t) = e^{-it:\hat{H}:}. \tag{7.20}$$

By comparing this formula with (7.10) one concludes that

$$Z(\beta) = \operatorname{Tr} \hat{U}(-i\beta) = \int d\varphi \, \langle\varphi|\hat{U}(-i\beta)|\varphi\rangle, \tag{7.21}$$

i.e. the partition function is obtained from (7.20) as a result of the substitution

$$t \to -i\beta. \tag{7.22}$$

Transformation (7.22) is called the Wick rotation. Taking the trace in (7.21) is equivalent to imposing periodic boundary conditions in the Euclidean "time" $\tau$ because one sums over all "transition amplitudes" $\langle\varphi|\hat{U}(-i\beta)|\varphi\rangle$ which start and end at the same configuration $\varphi$.

## 7.3 Effective Action and Free Energy

**Quantum Mechanics** There is a relation between effective action (7.2) and free energy (7.16), (7.17). To see this, we start with a quantum-mechanical model: a single Bose oscillator with the frequency $\omega$ being in a thermal state with the temperature $T = \beta^{-1}$. We define the effective action for the oscillator

$$W_+(\beta) \equiv -\frac{1}{2}\zeta'_+(0, \beta) \tag{7.23}$$

in terms of the following $\zeta$-function:

$$\zeta_+(s, \beta) \equiv \sum_{l=-\infty}^{\infty} \left[\sigma_l^2 + \omega^2\right]^{-s}. \tag{7.24}$$

One can show that $\zeta_+(s, \beta)$ is well-defined at $\Re s > 1/2$ and its value in the rest part of the complex plane can be obtained by analytical continuation. The numbers

$$\sigma_l = \frac{2\pi l}{\beta} \tag{7.25}$$

are called the Matsubara frequencies. It can be shown, see Exercise 7.7, that

$$W_+(\beta) = \ln\bigl(1 - e^{-\beta\omega}\bigr) + \frac{\beta\omega}{2} = \beta\bigl(F(\beta) + E_0\bigr), \tag{7.26}$$

where $E_0 = \omega/2$ and $F(\beta)$ is the free energy of the oscillator which follows from (7.18). On the other hand, the Ray-Singer formula (5.46) allows one to consider $W_+(\beta)$ as a determinant of an operator,

$$W_+(\beta) = \frac{1}{2}\ln\det\bigl(-\partial_\tau^2 + \omega^2\bigr),$$

provided that $-\partial_\tau^2 + \omega^2$ acts on a space of functions $\varphi(\tau)$ subject to the periodicity condition

$$\varphi(\tau + \beta) = \varphi(\tau). \tag{7.27}$$

The corresponding base manifold is a circle of the length $\beta$. It is easy to see that $W_+(\beta)$ is given in terms of a path integral like (7.3) with the classical action

$$I[\varphi] = \frac{1}{2}\int_0^\beta d\tau\bigl((\partial_\tau\varphi)^2 + \omega^2\varphi^2\bigr), \tag{7.28}$$

where $\varphi(\tau)$'s are periodic functions (7.27). Note that equation of motion in such classical theory, $\partial_\tau^2\varphi = \omega^2\varphi$, is related by the Wick rotation $t \to -i\tau$ to the equation of motion for a harmonic oscillator.

The case of a Fermi oscillator is considered analogously. One defines the effective action

$$W_-(\beta) = -\ln\bigl(1 + e^{-\beta\omega}\bigr) - \frac{\beta\omega}{2} = \beta\bigl(F(\beta) + E_0\bigr), \tag{7.29}$$

where $F(\beta) = -\beta^{-1}\ln(1+e^{-\beta\omega})$ is the free energy of a single Fermi degree of freedom with the frequency $\omega$. The vacuum energy $E_0 = -\omega/2$ differs from the vacuum energy of a boson by the sign, see Sect. 2.5 and Eq. (2.52). The r.h.s. of (7.29) is chosen to coincide with (7.26). One can prove that

$$W_-(\beta) = \frac{1}{2}\zeta'_-(0,\beta), \tag{7.30}$$

$$\zeta_-(s,\beta) = \sum_{l=-\infty}^{\infty}\left[\sigma_l^2 + \omega^2\right]^{-s}, \tag{7.31}$$

where the Matsubara frequencies are now defined as

$$\sigma_l = 2\pi\left(l + \frac{1}{2}\right)\frac{1}{\beta}. \tag{7.32}$$

Derivation of (7.30) is left as Exercise 7.7. The effective action can be rewritten by using the Ray-Singer formula,

$$W_-(\beta) = -\frac{1}{2}\ln\det(-\partial_\tau^2 + \omega^2) = -\ln\det(\partial_\tau + \omega), \tag{7.33}$$

where the operators act on a space of functions on the circle subject to the anti-periodic condition

$$\psi(\tau+\beta) = -\psi(\tau). \tag{7.34}$$

On the last line of (7.33) we used the fact that $\partial_\tau + \omega$ and $\partial_\tau - \omega$ have the same spectra (as a consequence of the symmetry of the Matsubara frequencies (7.32)). The operator $\partial_\tau + \omega$ is analogous to the Dirac operator $\not{D}$. The definition of the spectral function $\ln\det(\partial_\tau + \omega)$ follows the procedure developed in Sect. 5.6.

The difference between the Bose and Fermi statistics is not only in the boundary conditions. The functionals $W_+$ and $W_-$ have different signs by the logarithms of the determinants. For this reason $W_-$ is defined by the so-called Berezin path integral

$$e^{-W_-(\beta)} = \int D\bar{\psi}D\psi e^{-I[\bar{\psi},\psi]}, \tag{7.35}$$

$$I[\bar{\psi},\psi] = \int \bar{\psi}(\tau)(\partial_\tau + \omega)\psi(\tau)\,d\tau, \tag{7.36}$$

where $\psi(\tau)$ and $\bar{\psi}(\tau)$ are Grassmann functions which obey (7.34). The integration rules for a single degree of freedom are

$$\int d\psi = 0, \quad \int d\bar{\psi} = 0, \quad \int \psi\,d\psi = 1, \quad \int \bar{\psi}\,d\bar{\psi} = 1. \tag{7.37}$$

It is implied that the differentials $d\psi$ and $d\bar{\psi}$ are Grassmannian variables as well. It can be easily found

$$\int d\bar{\psi}\,d\psi\,e^{-a\bar{\psi}\psi} = \int d\bar{\psi}\,d\psi\,(1 - a\bar{\psi}\psi) = a. \tag{7.38}$$

This formula serves as a motivation for (7.35), as is shown in Exercise 7.2.

**Field Theory** Consider a free quantum field on a stationary background. Similar to the case of a single oscillator there is a relation between the free energy and the effective action in a field theory. To give an idea how this relation appears we use a complex scalar field model, as an example.

Equations of motion on stationary backgrounds take the form, see Sect. 2.5,

$$P(\partial_t, \partial_k)\varphi(t, x^k) = -(g^{\mu\nu}\partial_\mu\partial_\nu + a^\mu\partial_\mu + b(x))\varphi(t, x^k) \quad (7.39)$$

$$= \big(-P_0\partial_t^2 + P_1 i\partial_t + P_2\big)\varphi(t, x^k) = 0, \quad (7.40)$$

where $x^k$ are spatial coordinates and $P_p$ is a $p$-th order differential operator which does not depend on $t$. After the substitution $\varphi(t, x^k) = e^{-i\omega t}\varphi_\omega(x^k)$ in (7.39) one comes to nonlinear spectral problem (6.3)

$$P(i\omega, \partial_k)\varphi_\omega(x^k) = \big[-\omega^2 + L(\omega)\big]\varphi_\omega(x^k) = 0, \quad (7.41)$$

$$L(\omega) = L_2 + \omega L_1 + \omega^2 L_0. \quad (7.42)$$

Here $L_p$ is a $p$-th order operator. Examples of Eqs. (7.41), (7.42) were discussed in Sect. 6.1.

With the help of (7.16) and the eigenvalues $\omega_i$ of (7.41) one defines the free energy $F(\beta)$. The aim is to express $F(\beta)$ in terms of the corresponding effective action $W(\beta)$ which is the logarithm of determinant of some operator $P_E$. The experience with the harmonic oscillator suggests that $P_E$ after Wick rotation (7.22) can be associated with wave equation (7.40). Therefore, we propose, and then prove, the following relation:

$$W(\beta) = \ln\det P_E = -\zeta'(0; P_E), \quad (7.43)$$

where $P_E = P_E(\partial_\tau, \partial_k) \equiv P(i\partial_\tau, \partial_k)$. The periodic boundary conditions (7.27) with respect to coordinate $\tau$ are implied in (7.43). We also assume that $P_E$ does not have zero eigenvalues. Since $\varphi$ is supposed to be complex, the r.h.s. of (7.43) does not contain a factor of $1/2$.

In general, $P_E$ it is not self-adjoint due to terms which contain a single time derivative. It is however a Laplace type operator, which means that its heat kernel $K(P_E; t)$ is well-defined. The zeta-function $\zeta(s; P_E)$ is determined in terms of $K(P_E; t)$ by (5.24) despite the complex eigenvalues. Consider the eigenvalue problem

$$P_E\varphi_\Lambda = \Lambda\varphi_\Lambda. \quad (7.44)$$

Due to the isometry in $\tau$ and periodicity (7.27) one can make the substitution

$$\phi_\Lambda(\tau, x) = e^{i\sigma_l\tau}\phi_\Lambda(x), \quad (7.45)$$

where $\sigma_l$ are the Matsubara frequencies (7.25), and get with the help of (7.42) the following problem:

$$(\sigma_l^2 + L(i\sigma_l))\varphi_\Lambda(x) = \Lambda\varphi_\Lambda(x). \quad (7.46)$$

It is convenient to consider a related sequence of the eigenvalue problems motivated by (6.7)

$$L(i\sigma_l)\varphi_\lambda^{(\sigma_l)}(x) = \lambda(i\sigma_l)\varphi_\lambda^{(\sigma_l)}(x). \tag{7.47}$$

Then spectrum in (7.44) is $\Lambda_l = \sigma_l^2 + \lambda(i\sigma_l)$. We assume for simplicity that the spectrum of $\lambda(i\sigma_l)$ is discrete and consider the zeta-function

$$\zeta(s; P_E) = \varrho^{-2s} \sum_{\sigma_l} \sum_\lambda (\sigma_l^2 + \lambda(i\sigma_l))^{-s}. \tag{7.48}$$

Series (7.48) converges when $\Re s > n/2$, where $n$ is the number of space-time dimensions, $n \geq 2$. It is easy to see that $\lambda^*(i\sigma_l) = \lambda(-i\sigma_l)$ and, therefore, $\zeta(s; P_E)$ is real for real values of $s$ (as a consequence of the symmetry of the Matsubara spectrum, $-\sigma_l = \sigma_{-l}$). That is why $W(\beta)$ is real as well.

By using the Cauchy theorem one can rewrite (7.48) as

$$\zeta(s; P_E) = \frac{\varrho^{-2s}}{2\pi i} \sum_{\sigma_l} \sum_\lambda \int_C \frac{dz}{z - \sigma_l} (z^2 + \lambda(iz))^{-s}. \tag{7.49}$$

The contour $C$ consists of two parallel lines, $C_+$ and $C_-$, in the complex plane. $C_+$ goes from $(i\epsilon + \infty)$ to $(i\epsilon - \infty)$ and $C_-$ goes from $(-i\epsilon - \infty)$ to $(-i\epsilon + \infty)$. Summation over $\sigma_l$ in (7.49) can be performed with the help of (4.128) (see Exercise 4.9)

$$\zeta(s; P_E) = \frac{\varrho^{-2s}\beta}{4\pi i} \sum_\lambda \int_C dz \cot\left(\frac{\beta z}{2}\right)(z^2 + \lambda(iz))^{-s}. \tag{7.50}$$

For the integral over $C_+$ we use the identity

$$\cot\left(\frac{\beta z}{2}\right) = \frac{2}{\beta}\frac{d}{dz}\ln\left(1 - e^{i\beta z}\right) - i,$$

while for $C_-$

$$\cot\left(\frac{\beta z}{2}\right) = \frac{2}{\beta}\frac{d}{dz}\ln\left(1 - e^{-i\beta z}\right) + i.$$

One can change $z$ to $-z$ on $C_-$ to get

$$\zeta(s; P_E) = \beta f_1(s) + f_2(s; \beta), \tag{7.51}$$

$$f_1(s) = \frac{\varrho^{-2s}}{2\pi} \sum_\lambda \int_{-\infty}^{\infty} (x^2 + \lambda(ix))^{-s}\, dx, \tag{7.52}$$

$$f_2(s; \beta) = \frac{\varrho^{-2s}}{2\pi i} \sum_\lambda \int_{C_+} dz \frac{d}{dz}\ln\left(1 - e^{i\beta z}\right)\left[(z^2 + \lambda(iz))^{-s} + (z^2 + \lambda(-iz))^{-s}\right]. \tag{7.53}$$

The function $f_2(s;\beta)$ represents a purely 'thermal part' of the zeta-function. It vanishes at zero temperature because of exponentially small factor coming from $e^{i\beta z}$.

The contour $C_+$ in (7.53) can be deformed to make the integrand exponentially small at large $z$ due to the factor $e^{i\beta z}$. One can integrate then by parts to get

$$f_2(s;\beta) = s\frac{\varrho^{-2s}}{2\pi i}\sum_\lambda \int_{C_+} dz \ln\big(1-e^{i\beta z}\big)\left(\frac{\partial_z\breve{\chi}(z,\lambda)}{(z^2+\lambda(iz))^{s+1}} + \frac{\partial_z\breve{\chi}(-z,\lambda)}{(z^2+\lambda(-iz))^{s+1}}\right), \tag{7.54}$$

where $\breve{\chi}(z,\lambda) \equiv z^2 + \lambda(iz)$. With the help of (7.43) and (7.51) the effective action is expressed as

$$W(\beta) = -\big(f_2'(0;\beta) + \beta f_1'(0)\big). \tag{7.55}$$

To proceed one notes that $f_2(s;\beta)$ has a form $sg(s;\beta)$ where $g(s;\beta)$ is a function which is finite at $s=0$. Hence, $f_2'(0;\beta) = g(0;\beta)$, see (7.54). To compute $g(0;\beta)$ one adds to $C_+$ a large semicircle lying in the upper half of the complex plane and makes a closed contour. The exponent $e^{i\beta z}$ in the logarithm in (7.54) guarantees that integration over the semicircle vanishes at large radii. The Cauchy theorem then implies that

$$-f_2'(0;\beta) = \sum_{z_+}\ln\big(1-e^{\beta i z_+}\big) + \sum_{z_-}\ln\big(1-e^{\beta i z_-}\big), \tag{7.56}$$

where $z_\pm$ are roots of algebraic equations

$$z_\pm^2 + \lambda(\pm i z_\pm) = 0. \tag{7.57}$$

(One should choose the roots in the upper part of the complex plane, $\Im\, z_\pm > 0$.) Let us return to Eqs. (6.8), (6.9) for the spectrum of physical single-particle energies $w_i$ in case of non-linear spectral problems. If we put $z = i\omega$, Eq. (7.57) becomes equivalent to (6.8). On physical grounds we assume that all frequencies $\omega_i$ are real, therefore, all roots of (7.57) lie on the imaginary axis. One concludes that

$$-f_2'(0;\beta) = \sum_{w_i}\ln\big(1-e^{-\beta w_i}\big) = \beta F(\beta), \tag{7.58}$$

where $F(\beta)$ is the free energy of the system. The sum in (7.58) includes both the positive and negative norm modes (the corresponding energies are $w_i^{(+)}$ and $w_i^{(-)}$) discussed in Sect. 2.1. For a real field, $\lambda(iz) = \lambda(-iz)$ and the two terms in (7.54) coincide.

According to (7.55) the relation between $W(\beta)$ and the free energy is the same as in the case of the harmonic oscillator,

$$W(\beta) = \beta(F(\beta) + E_0). \tag{7.59}$$

The constant $E_0$,

$$E_0 = -f_1'(0) \tag{7.60}$$

has the meaning of the vacuum energy. We leave the proof of this fact for Exercise 7.10.

Let us emphasize that problem (7.41) is uniquely defined by the wave operator (7.40) but the opposite statement is not true. It is clear, for example, that the wave operators $P(\partial_t, \partial_k)$ and $P'(\partial_t, \partial_k) = f(x)P(\partial_t, \partial_k)$ (where $f(x)$ is a non-degenerate function of spatial coordinates) describe the same single-particle spectrum $\omega_i$. Therefore, the theories determined by the operators $P$ and $P'$ have the same free energy. The effective actions of these theories, however, differ by finite terms. This happens because there is a non-trivial transformation from $\ln\det P_E$ to $\ln\det P'_E$ due to the properties of the Ray-Singer determinants considered Sect. 5.7. It follows from (7.59) that the vacuum energies for theories determined by $P$ and $P'$ differ as well.

We have analyzed a scalar field model. The results can be extended to the case of Dirac fermions. The free energy in this case is given by (7.17) and can be related by (7.59) to the effective action $W(\beta) = -\ln\det \not{D}$. This is an extension of Eqs. (7.29) and (7.33) for a single Fermi oscillator.

## 7.4 Complex Geometries

To summarize the previous section, the free energy in a finite temperature field theory can be determined in terms of an effective action which is the Ray-Singer determinant for some class of Laplace type operators $P_E$, see (7.43). The information about the quantum state appears in periodic or anti-periodic boundary conditions in temporal coordinate $\tau$ which has the period $\beta = 1/T$. For the purposes of the present section it is convenient to use the notation $\phi_E$ for background fields and $W[\phi_E]$ for the effective action.

The Green's function for the operator $P_E$ is expressed in terms of the thermal averages

$$G(x, x') = \theta(\tau)\langle(\hat{\varphi}_E(x)\hat{\varphi}_E^+(x'))\rangle_\beta + \theta(-\tau)\langle(\hat{\varphi}_E^+(x')\hat{\varphi}_E(x))\rangle_\beta, \tag{7.61}$$

where $x = (\tau, y^k)$, $x' = (0, (y')^k)$ and

$$\hat{\varphi}_E(\tau, y) \equiv \hat{\varphi}(-i\tau, y), \tag{7.62}$$

see Exercises 7.12–7.14. One can immediately conclude, based on arguments of Sect. 7.1, that variations of the effective action over $\phi_E$ yield values of certain operators in the thermal state.

To proceed with this statement we should dwell on properties of background fields. The operator $P_E$ can be brought to the following form:

$$P_E = -(g_E^{\mu\nu}\partial_\mu\partial_\nu + a_E^\mu\partial_\mu + b(x)), \tag{7.63}$$

see (3.1). $P_E$ is obtained from a "Lorentzian" operator $P$ under the Wick rotation $t \to -i\tau$. By the construction, coefficients in (7.63) are related to coefficients in (7.39) as

$$g_E^{\tau\tau} = -g^{tt}, \quad g_E^{\tau k} = ig^{tk}, \quad g_E^{ik} = g^{ik}, \quad a_E^\tau = ia^t, \quad a_E^k = a^k. \tag{7.64}$$

To understand better the physical meaning of these quantities one can introduce a functional $I_E[\varphi, \phi_E] \equiv (\varphi^+, P_E\varphi)$ which is a Euclidean analog of the classical action. The dynamical variables $\varphi$ in $I_E[\varphi, \phi_E]$ are defined on some complex manifold $\mathscr{M}_E$ with a metric $(g_E)_{\mu\nu}$ which is the inverse matrix of $g_E^{\mu\nu}$. Evidently, the line element on $\mathscr{M}_E$ is obtained from the metric $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ on the corresponding physical Lorentzian space-time $\mathscr{M}$ under the Wick rotation. If the theory contains the gauge connections determined by the vector $a^\mu$ their time components have to be complexified as well, in accord with (7.64).

One concludes that a finite-temperature theory is equivalent to a covariant theory on a *complex* background $\phi_E$. There is a simple rule which relates a physical stationary background $\phi$ to its complex counterpart $\phi_E$: the complex fields (the metric, gauge connections and etc.) are obtained from the physical fields by multiplying their components by the factor $i^{q-p}$ where $q$ and $p$ are, respectively, the number of upper and lower temporal indexes.

Consider variations of $W[\phi_E]$ with respect to the Euclidean metric,

$$T_E^{\mu\nu}(x) = \frac{2}{\sqrt{\det|g_E|}} \frac{\delta W[\phi_E]}{\delta(g_E)_{\mu\nu}(x)}. \tag{7.65}$$

The arguments of Sect. 7.1 allow one to relate $T_E^{\mu\nu}$ to the thermal average of the components of the stress-energy tensor,

$$T_E^{\tau\tau} = -\langle \hat{T}^{tt}\rangle_\beta, \qquad T_E^{\tau k} = i\langle \hat{T}^{tk}\rangle_\beta. \tag{7.66}$$

Expectation values of other operators can be defined analogously.

## 7.5 Renormalization

The effective action method is equivalent to computations with the help of the point-splitting procedure. There is, however, an apparent contradiction because the averages in the point-splitting procedure contain divergent terms which arise from the singularities in Green's functions, see Sect. 2.7.

The fact that effective action (7.43) appears to be free from the divergences is an artifact of the $\zeta$-function regularization. A complete structure of the divergences in quantities like $\ln\det P_E$ can be studied with the help of PTC regularization considered in Sect. 5.8. By using (5.74) one can find the divergent part, $W_{\mathrm{div}}$, of the effective action,

$$W_{\mathrm{div}}[\phi_E, \delta] = 2\sum_{p=0}^{n-1} \frac{a_p(P_E)}{p-n}\delta^{\frac{p-n}{2}} + (a_n(P_E) - N)\ln\delta. \tag{7.67}$$

Here $\delta$ is a regularization parameter, $N$ is the number of zero modes of $P_E$, and $a_p(P_E)$ are the heat kernel coefficients of $P_E$. Variations of $W_{\mathrm{div}}$ over the metric yield the divergent part of stress-energy tensor (7.65), in agreement with the point-splitting method.

Let us briefly describe the renormalization procedure which enables one to get rid of the divergences and get physical quantities. Consider, as an example, the Einstein equations (1.21) and assume that the quantum matter is the only source of the gravitational field. One can cast the equations in a slightly different form,

$$\frac{1}{8\pi G_N}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)+\frac{\Lambda}{8\pi G_N}g_{\mu\nu}=\langle\hat{T}_{\mu\nu}\rangle_\beta. \tag{7.68}$$

The idea of renormalization is that the geometrical structure of the leading divergent terms in $\langle\hat{T}^{\mu\nu}\rangle_\beta$ coincides with the structure of the geometrical terms on the left hand side of (7.68). Thus, one can redefine (renormalize) the coupling constants, such as $G_N$ and $\Lambda$, to absorb divergences.

This procedure is more convenient to carry out on the level of the effective action. Consider a functional

$$\Gamma[g_E]=I_B[g_E]+W[g_E]. \tag{7.69}$$

Here $I_B[g_E]$ is some classical action which has a pure geometrical form. It includes the Einstein action and some higher curvature terms. For instance, in a four-dimensional theory, $n=4$, on a closed manifold

$$I_B[g_E]=I[g_E;G_B,\Lambda_B,c_B^i]=\int d^4x\,\sqrt{g_E}L, \tag{7.70}$$

$$L=\left[-\frac{\Lambda_B}{8\pi G_B}-\frac{R}{16\pi G_B}+c_B^1R^2+c_B^2R_{\mu\nu}R^{\mu\nu}+c_B^3R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\right]. \tag{7.71}$$

(The difference in the sign with respect to the gravity action in the Lorentzian theory (1.20) is related to the Wick rotation.) The constants $G_B,\Lambda_B,c_i$ are called the "bare" constants. Coefficients $a_p(P_E)$ in (7.67) are integrals of curvature powers on $\mathscr{M}_E$, see (4.30)–(4.32). The quadratic curvature terms in (7.71) are needed to eliminate the logarithmic divergences determined by $a_4(P_E)$. The logarithmic term $N\ln\delta$ is non-local. It reflects the presence of infrared divergences, which have to be treated in a different way.

The renormalization prescription consists in adding the divergent part $W_{\rm div}[g_E,\delta]$ to the bare classical action and redefining the couplings

$$I[g_E;G_B,\Lambda_B,c_B^i]+W_{\rm div}[g_E,\delta]=I[g_E;G_N,\Lambda,c^i], \tag{7.72}$$

$$W_{\rm ren}[g_E]=W[g_E]-W_{\rm div}[g_E,\delta]. \tag{7.73}$$

The constants $G_N$, $\Lambda$, $c^i$ are identified with the physical classical couplings while the rest part $W_{\rm ren}[g_E]$ (renormalized effective action) is used for calculation of quantum corrections. For example, a quantum non-minimally coupled scalar field with the operator $P_E=-\nabla^2+\xi R+m^2$ results in the following relation between the bare and physical Newton constants:

$$\frac{1}{G_N}=\frac{1}{G_B}+\frac{1}{2\pi\delta}\left(\frac{1}{6}-\xi\right). \tag{7.74}$$

The physical value of $G_N$ is well measured. The precise values of the cosmological constant $\Lambda$ and couplings $c_i$ are a matter of future physical tests.

## 7.6 Mean Field and the Coleman-Weinberg Potential

The effective action can be used to calculate averages of operators. In particular, one can use this method to find the average value of a dynamical variable itself. Consider, as an example, a model of a real scalar field in four-dimensional space-time ($n = 4$). The corresponding action (after the Wick rotation) is

$$I[\varphi] = \int d\tau\, d^3x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + U(\varphi)\right], \tag{7.75}$$

where $U(\varphi)$ is a classical potential. The space-time is flat and the field is contained in a box of a large volume $V$. The potential $U(\varphi)$ is assumed to be an even polynomial of $\varphi$ such that $U''(\varphi) > 0$.

Let us write $\varphi(x)$ in the following form

$$\varphi(x) = \phi + \chi(x) \tag{7.76}$$

and require that

$$\int_V d^3x\, \chi(x) = 0. \tag{7.77}$$

Here $\phi$ is a constant which can be defined (in accord with (7.77)) as

$$\phi = \frac{1}{V}\int_V d^3x\, \varphi(x). \tag{7.78}$$

Hence, $\phi$ is the average value of the classical field configuration in the volume $V$. In what follows we identify $\phi$ with a quantum average of the field operator, the so-called mean field. Deviations $\chi(x)$ are related to quantum fluctuations. The configuration $\phi$ serves as a classical background for $\chi(x)$.

Suppose that the system considered is in a thermal equilibrium with the temperature $T = \beta^{-1}$. The equilibrium state implies that $\phi$ does not depend on time. To use the results of the previous sections we need a linear approximation in the theory. This is possible when excitations are small and their self-interactions can be neglected as compared to the interaction with the background. In this regime classical action (7.75) can be approximated by the expression

$$I[\phi + \chi] = \beta V U(\phi) + I_1[\chi, \phi] + O(\chi^3), \tag{7.79}$$

$$I_1[\chi, \phi] = \frac{1}{2}\int d^4x \left[\partial_\mu\chi\partial^\mu\chi + m^2(\phi)\chi^2\right], \tag{7.80}$$

$$m^2(\phi) = U''(\phi). \tag{7.81}$$

In the given approximation the total action can be written as

$$\Gamma[\phi] = V\beta U(\phi) + W[\phi], \tag{7.82}$$

where $W[\phi]$ is the effective action of the excitations

$$W[\phi] = \frac{1}{2}\ln\det P_E[\phi]. \tag{7.83}$$

The operator $P_E = \varrho^2(-\partial^2 + m^2(\phi))$ is defined on the base manifold $S^1 \times R^3$, the length of $S^1$ being equal to $\beta$. A dimensional parameter $\varrho$ is introduced to make the operator dimensionless.

In the considered approximation $\chi$ is a non-interacting field and its average vanishes, $\langle\hat{\chi}\rangle_\beta = 0$, see Exercise 7.15. As a consequence, $\langle\hat{\varphi}\rangle_\beta = \phi$. What is left to determine is the value of $\phi$. It is natural to require that $\phi$ coincides with a minimum, $\phi_0$, of the total action

$$\left.\frac{\partial\Gamma[\phi]}{\partial\phi}\right|_{\phi=\phi_0} = 0. \tag{7.84}$$

The requirement comes from the classical theory. Also in statistical physics the equilibrium state corresponds to a minimum of the free energy.

The divergent part $W_{\text{div}}[\phi,\delta]$ of (7.83) is determined by (7.67),

$$W_{\text{div}}[\phi,\delta] = \frac{\beta V}{16\pi^2}\left(-\frac{1}{4\delta^2\varrho^4} + \frac{m^2(\phi)}{2\delta\varrho^2} + \frac{(m^2(\phi))^2}{4}\ln\delta\right). \tag{7.85}$$

Since $P_E$ is dimensionless, so does the regularization parameter $\delta$. The physical high-energy cutoff is $\delta^{-1/2}\varrho^{-1}$. There is a mismatch in the factor $1/2$ in (7.85) with respect to (7.67) because the field is real. Boundary terms in (7.85) are neglected because the volume $V$ is large.

The renormalization can be carried out if $W_{\text{div}}[\phi,\delta]$ has the same structure as the classical action $I[\phi] = \beta V U(\phi)$. Let $U(\phi)$ be a $N$-th order polynomial in $\phi$. According to (7.81) and (7.85), the divergent part $W_{\text{div}}[\phi,\delta]$ is a polynomial of the order $2(N-2)$. Renormalization condition requires that $2(N-2) \le N$ or $N \le 4$. This implies the following form for the potential:

$$U(\phi) = \frac{a}{2}\phi^2 + \frac{b}{12}\phi^4, \tag{7.86}$$

where $a$ and $b$ are some constants. We omit an additive constant in the potential $U(\phi)$ because it does not change its minimum. Once the divergences are renormalized away, one can use the Ray-Singer formula for a finite part of effective action (7.83)

$$W[\phi] = -\frac{1}{2}\zeta'(0; P_E). \tag{7.87}$$

We have to stress that the divergent part of the effective action is defined up to a finite expression having the same structure as $W_{\text{div}}$. Consequently, the same expression may appear with an opposite sign in (7.87). This ambiguity corresponds to a *finite renormalization of couplings* and has to be removed by imposing the so-called normalization conditions discussed in Exercise 7.16.

Our primary interest is the vacuum state which is a limit of vanishing temperature ($\beta \to \infty$). In this limit, because the volume $V$ is large, the heat kernel can be approximated as $K(P_E;t) \simeq (V\beta/\varrho^4)\exp(-tm^2\varrho^2)/(4\pi t^2)^2$. This yields for the zeta-function

$$\zeta(s;P_E) = \frac{\beta V}{16\pi^2\varrho^4}\frac{(m^2\varrho^2)^{2-s}}{(s-1)(s-2)}. \tag{7.88}$$

It is convenient to introduce a density $\Omega[\phi]$ of action (7.82),

$$\Omega[\phi] = \Gamma[\phi]/(\beta V). \tag{7.89}$$

With the help of (7.86)–(7.88) one finds

$$\Omega(\phi) = U(\phi) + \frac{1}{64\pi^2}\big(U''(\phi)\big)^2 \ln\big(\varrho^2 U''(\phi)\big). \tag{7.90}$$

It is implied that $U''(\phi) > 0$. Derivation of (7.90) ignores the condition (7.77). The only eigenfunction of $P_E$ which does not respect (7.77) is a constant mode with the eigenvalue $m^2(\varphi)$. The contribution of this mode has to be excluded from the spectrum. The corresponding modification of the potential is not essential, however, in the limit of large volume.

The function $\Omega(\phi)$ is an important object called the *Coleman-Weinberg potential*. The Coleman-Weinberg potential includes the classical part $U(\phi)$ given by (7.86) in terms of the renormalized constants $a$, $b$ and a quantum correction. Minimal points of $\Omega(\phi)$ determine the values of the mean field in the quantum theory. Properties of the Coleman-Weinberg potential are further studied in the exercises to this Chapter.

## 7.7 Feynman Diagrams and Beta Functions

The notion of the effective action goes beyond the scope of non-interacting finite-temperature theories. The effective action can be introduced in the presence of interactions and it acquires an additional meaning as a generating functional for a certain class of Green's functions, so-called one-particle irreducible diagrams. Some effects related to interactions can be described even with techniques introduced in the previous sections. These effects are reduced to interactions between dynamical and background fields.

To illustrate this idea consider the effective action in quantum electrodynamics (QED). Due to the presence of virtual electron-positron pairs the electric potential of a test particle in a vacuum state in QED differs from the Coulomb form. Modification of the Coulomb's law can be taken into account by adding to the classical Maxwell functional the effective action of an electron in a background electric field with potential $A_\mu$

$$W[A] = -\ln \det{}_\varrho(\not{D}(A) + m). \tag{7.91}$$

Here $\not{D}(A) = i\gamma^\mu D_\mu(A)$, $D_\mu(A) = \partial_\mu + ieA_\mu$, $m$ is the mass of the electron and $e$ is its charge. The action is defined by the corresponding determinant in a gauge background. It is a variation of $W[A]$ over $A_\mu$ that yields a quantum correction to the Maxwell equations.

For a computational convenience we use finite-temperature formalism and assume that $A_\mu$ is static, though little changes in the calculations for a generic form of the background electromagnetic potential. The vacuum case is restored in the limit of zero temperature. In the PTC regularization the action is

$$W[A] = -\frac{1}{2}\ln\det\varrho^2(L(A)+m^2) = \frac{1}{2}\int_\delta^\infty \frac{dt}{t}K(L(A);t)e^{-tm^2}, \quad (7.92)$$

$$L(A) = \not{D}^2(A) = -D^\mu(A)D_\mu(A) + \frac{ie}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}. \quad (7.93)$$

We have rescaled in (7.92) the dimensionless PTC cutoff parameter by multiplying it by the factor $\varrho^2$. From now on $\delta$ has the dimension of the length square.

The strength of the electromagnetic field is supposed to be weak, $|eF_{\mu\nu}| \ll m^2$, where $F_{\mu\nu}$ is the Maxwell tensor. Under these conditions, because of the factor $e^{-tm^2}$, the main contribution to (7.92) comes at small $t$ ($\delta < t < m^{-2}$). One can use the asymptotic expansion for $K(L(A);t)$ to get

$$W[A] \simeq \frac{1}{2}\sum_{p=0} a_p(m^2)^{2-p/2}\Gamma\big(p/2-2, m^2\delta\big). \quad (7.94)$$

Here $a_p$ are the heat coefficients of $L(A)$, and

$$\Gamma(z,x) = \int_x^\infty dy\, y^{z-1}e^{-y}$$

is the incomplete gamma-function, $x > 0$. We neglect boundary effects and use a short $x$ asymptotic of $\Gamma(z,x)$ to get the following expansion:

$$W[A] \sim C + \frac{a_4}{2}\ln m^2\delta + \frac{a_6}{2m^2} + \frac{a_8}{2m^4} + \cdots. \quad (7.95)$$

We have noted that $a_2 = 0$, see (4.57). The coefficient $a_0$ yields a constant divergent contribution $C$ which is not essential for the further discussion. It follows from (4.58) that

$$a_4 = \frac{e^2}{24\pi^2}\int d^4x\, F_{\mu\nu}F^{\mu\nu}, \quad (7.96)$$

where $\int d^4x = \beta\int d^3x$. It is not difficult to understand that each coefficient $a_p$ is a polynomial in the fine-structure constant $\alpha \equiv e^2/(4\pi)$, the order of the polynomial being $[p/4]$, $p$ is even. Therefore, (7.94) can be also rearranged as an expansion in $\alpha$. One may say that (7.94) sums contributions of what is known as the one-loop Feynman diagrams. Each such a diagram consists of a fermion loop and a number of lines corresponding to $A_\mu$, called the "external legs". Examples of one-loop diagrams are shown on Fig. 7.1. A diagram proportional to $\alpha^k$ has the following structure:

$$\begin{aligned}&\int d^4x_1\, d^4x_2 \ldots d^4x_{2k}\, W_{(2k)}^{\mu_1,\mu_2,\ldots,\mu_{2k}}(x_1,x_2,\ldots,x_{2k})A_{\mu_1}(x_1)\ldots A_{\mu_{2k}}(x_{2k})\\ &\quad = (-1)^k e^{2k}\int d^4x_1\, d^4x_2 \ldots d^4x_{2k}\,\mathrm{Tr}\big[\gamma^{\mu_1}G(x_1-x_2)\gamma^{\mu_2}G(x_2-x_3)\ldots\\ &\qquad \ldots\gamma^{\mu_{2k}}G(x_{2k}-x_1)\big]A_{\mu_1}(x_1)\ldots A_{\mu_{2k}}(x_{2k})\end{aligned} \quad (7.97)$$

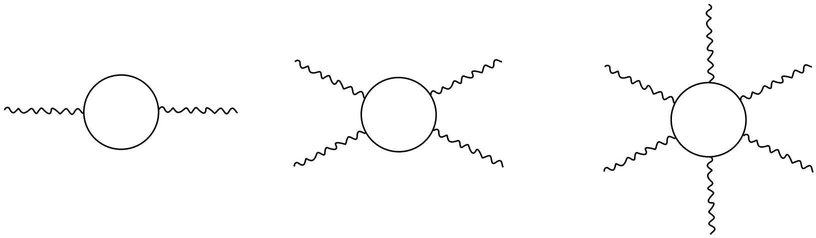

**Fig. 7.1** Feynman diagrams which contribute to the effective action in QED. The loops correspond to propagation of virtual electrons and positrons, the legs are the photons. The vertexes describe interaction of virtual particles with the external electromagnetic background. Each vertex carries the electric charge $e$

where $G = (\not{D}(0) + m)^{-1}$ is the Green function for the Dirac spinor, see the Exercise 7.22. Our results follow in the limit of large $m$. Diagrams with odd number of legs vanish in agreement with the symmetry of the effective action (7.91), $W[A] = W[-A]$. This property constitutes a statement of the Furry theorem.

Heat kernel coefficient (7.96) is related to a diagram on Fig. 7.1 with two legs. This diagram determines the so-called polarization operator and results in modification of the Coulomb's law. It is the only one-loop diagram which is divergent in four dimensions. The coefficient $a_8$ is proportional to the integral of $(F_{\mu\nu}F^{\mu\nu})^2$ and corresponds to a finite diagram with four legs. This diagram determines photon-photon scattering.

Let us discuss renormalization of the divergences. The total action in a static gauge field is the sum of classical and quantum parts (compare with (7.82))

$$\begin{aligned}\Gamma[A] &= \frac{1}{4}\int d^4x\ F_{\mu\nu}F^{\mu\nu} + \hbar W[A] \\ &= \frac{Z(\delta)}{4}\int d^4x\ F_{\mu\nu}F^{\mu\nu} + \hbar\frac{a_6}{2m^2} + \hbar\frac{a_8}{2m^4} + \cdots, \end{aligned} \tag{7.98}$$

$$Z(\delta) = 1 + \hbar\frac{e^2}{12\pi^2}\ln m^2\delta. \tag{7.99}$$

Here we have restored the Planck constant $\hbar$. To remove the divergences one defines a new gauge potential $\bar{A}_\mu = \sqrt{Z}A_\mu$. After that the first term in the effective action takes the canonical form,

$$\frac{Z(\delta)}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} = \frac{1}{4}\int d^4x\, \bar{F}_{\mu\nu}\bar{F}^{\mu\nu}, \tag{7.100}$$

where $\bar{F}_{\mu\nu}$ is the strength for $\bar{A}_\mu$. Because we restrict ourselves to the effects up to first order in $\hbar$ one can just replace $A_\mu$ by $\bar{A}_\mu$ in the next terms in (7.98).

If charged particles are present, there must be an interaction term in the Lagrangian, $eA_0\rho$, where $\rho$ is a density of the particles. To preserve the form of the interaction, the field redefinition should be accompanied with a renormalization of the charge, $\bar{e} = e/\sqrt{Z}$, and of the fine-structure constant. One gets the relation

$$\frac{1}{\bar{\alpha}} = \frac{4\pi}{\bar{e}^2} = \frac{Z(\delta)}{\alpha} = \frac{1}{\alpha} + \frac{\hbar}{3\pi}\ln m^2\delta. \tag{7.101}$$

Here $\bar{\alpha}$ is identified with an observable coupling, $\alpha$ is the bare constant.

We have calculated the divergent part of effective action only. The finite part, which generates the amplitudes, is a more complicated non-local functional of the background fields. However, some of the amplitudes may be viewed upon as classical ones but with the charges depending on a characteristic momenta $\mu$ of the particles participating in these amplitudes. In QED such running coupling constant is $\alpha(\mu^2)$. The rate of change of $\alpha(\mu^2)$ is measured by the so-called beta function

$$\beta(\alpha) = \mu^2 \frac{d\alpha(\mu^2)}{d\mu^2}. \tag{7.102}$$

It is an amazing property of some renormalizable field theories, including QED, that the beta function can be calculated from the same formula (7.101) with the replacement $\alpha(\mu^2) = \bar{\alpha}(1/\delta)$ where $\mu^2 = 1/\delta$, i.e. by using the cut-off instead of the energy scale. This fact, related to the lack of dimensionful parameters in an asymptotic regime, will be left here without further comments. The interested reader can consult textbooks mentioned at the end of this Chapter. In QED at the one-loop order the formula (7.101) with $\mu = 1/\sqrt{\delta}$ yields

$$\beta(\alpha) = \hbar \frac{\alpha^2}{3\pi}. \tag{7.103}$$

Positivity of the beta-function indicates that in QED the coupling grows with energies.

## 7.8 Gauge Fields and Ghosts

Let us discuss, in a rather non-rigorous manner, some generic features of the effective action of quantized gauge fields. To set the stage we first consider a simple example and return to results of Sect. 2.3.

The example is the free energy of photons in a pure Maxwell theory in a static gravitational field. To define the free energy one needs the single-particle spectrum of physical modes. One can use results of Exercises 2.5, 2.6, 2.9. In the Lorentz gauge $\nabla^\mu A_\mu = 0$ the Maxwell equations are reduced to

$$\nabla^2 A_\mu - R^\nu_\mu A_\nu = 0, \tag{7.104}$$

where $R^\nu_\mu$ is the Ricci tensor of the background metric. Let $\omega_i^{(1)}$ be the single-particle spectrum (vector modes) for Eq. (7.104), and $\omega_i^{(0)}$ be the corresponding spectrum (scalar modes) for scalar equation $\nabla^2\varphi = 0$ on the same background. It follows from the results of Exercise 2.9 that the free energy of photons is represented in the following form:

$$F(\beta) = F_{(1)}(\beta) - 2F_{(0)}(\beta), \tag{7.105}$$

where $F_{(1)}(\beta)$ and $F_{(0)}(\beta)$ are defined by (7.16) for the vector and scalar modes, respectively. Subtracting the contribution of the scalar modes in (7.105) is needed because the spectrum of vector modes which are solutions to (7.104) is larger than the physical spectrum. To satisfy the Lorentz gauge condition, one has to exclude the longitudinal modes for which $\nabla^\mu A_\mu \neq 0$ and the modes $A_\mu = \partial_\mu \lambda$ with $\nabla^2 \lambda = 0$. An analogous representation, $E_0 = E_0^{(1)} - 2E_0^{(0)}$, can be written for the vacuum energy of photons.

Let us note that $F(\beta)$ is a gauge-independent object. One can use other gauge fixing conditions leading to different functionals $F_{(1)}(\beta)$ and $F_{(0)}(\beta)$. The functional $F(\beta)$ remains unchanged.

An interesting result for the effective action of photons follows from (7.105). If one takes into account its relation with the free energy, Eq. (7.59), the action acquires the form

$$W(\beta) = \frac{1}{2} \ln \det(\Delta^{(1)}) - \ln \det(\Delta^{(0)}), \tag{7.106}$$

where $\Delta^{(1)}$ and $\Delta^{(0)}$ are vector and scalar Laplacians (3.5) and (3.4), respectively. One should pay attention to the last term in the r.h.s. of (7.106) where the determinant of the scalar Laplace operator enters with a wrong sign. Quantum fields which could be responsible for this term in the action should transform as scalars but quantized according with the Fermi statistics. Since such fields are unphysical they are called ghosts, namely the *Faddeev-Popov ghosts*.

Although we have started on a static background, formula (7.106) is valid on an arbitrary Euclidean space. In this generic case, the argument the effective action $W$ depends upon is not the inverse temperature $\beta$, but rather the whole $n$-dimensional geometry.

The appearance of ghosts does not mean, of course, that the theory is pathological. The ghosts are needed to keep the right number of physical degrees of freedom and to carry out the computation in a convenient way. Related to the existence of ghosts there is a special type of a global symmetry which plays an important role in quantizing the gauge theories. To see it let us write a classical action whose quantization leads to (7.106)

$$I[A, c, \bar{c}] = \int d^n x \sqrt{g} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\nabla A)^2 + \nabla_\mu \bar{c} \nabla^\mu c \right], \tag{7.107}$$

where $c$ and $\bar{c}$ are the ghost and anti-ghost fields. The functional is invariant with respect to the so-called Becchi-Rouet-Stora-Tyutin (or BRST) transformation

$$\delta A_\mu(x) = \nabla_\mu(\lambda c(x)), \qquad \delta \bar{c}(x) = -\lambda \nabla A(x), \qquad \delta c = 0, \tag{7.108}$$

where $\lambda$ is a Grassmann variable which anti-commutes with the ghost fields.

What we have just revealed is a particular application of a general approach to quantization of gauge theories. In fact, the approach has been suggested to deal with non-Abelian gauge fields. Consider an action $I[A]$ for some gauge fields $A$ which is invariant under infinitesimal gauge transformations $\delta_\xi A = l(\xi)$ where $\xi$ is a local parameter, and $l$ is a linear operator which can depend on $A$ or on other fields. Because of the gauge symmetry $I[A + \delta_\xi A] = I[A]$ the action is degenerate, and so is the relativistic inner product constructed from a linearized version of this action, see Sect. 2.3. To eliminate this degeneracy, let us introduce a set of gauge-fixing conditions $\mathscr{F}(A)=0$. These conditions eliminate the gauge freedom meaning that their solutions intersect each orbit of the gauge group in exactly one point. Generically, such conditions cannot be chosen globally on the whole space of the fields, but as we have already pointed out, since we are working with small fluctuations only, this is not a problem. Clearly, we need, roughly speaking, one gauge fixing conditions for each gauge transformation. Then one adds a gauge-breaking term $\frac{1}{2}\int d^n x\,\sqrt{-g}(\mathscr{F}(A))^2$, where one must include a summation if there is more than one condition per point. Also, one has to introduce ghost fields $\bar{c}$ and $c$ with the Lagrangian density $\bar{c}\mathscr{F}(l(c))$, so that the total action becomes

$$I_{\text{gauge fixed}}[A,c,\bar{c}]=I[A]+\int d^n x\,\sqrt{g}\left[\frac{1}{2}(\mathscr{F}(A))^2+\bar{c}\mathscr{F}(l(c))\right],\quad (7.109)$$

where the ghosts have the same transformation properties with respect to the space-time symmetries as the gauge transformation parameters, but are Grassmann variables. After the gauge freedom has been fixed, the resulting action is non-degenerate and can be quantized in the usual way, though with the fields having a non-standard statistics. One can easily see that action (7.107) has precisely the form (7.109).

The situation is not always so easy. In many models one has to introduce higher (cubic, quartic, etc.) ghost terms in the action as well as more ghosts fields (the so-called "ghosts for ghosts"). However, the very simple formalism described above works for most of the models in the leading (one-loop) order of the perturbation theory. This all is, of course, well known, and is included here mostly for the reference purposes. An important and less trivial application of the spectral theory in case of non-Abelian gauge models is considered in the next section.

## 7.9 The Asymptotic Freedom in Quantum Chromodynamics

We complete this Chapter with a computation of the one-loop beta function in the quantum chromodynamics (QCD). The computation is rather challenging in terms of Feynman diagrams and our purpose is to demonstrate here a full power of an alternative technique based on the spectral theory.

Before starting actual calculations, we have to make an important comment. The principal role of effective action in quantum field theory is to generate Feynman diagrams through variational derivatives with respect to background fields. Strictly speaking, only the so-called one-particle-irreducible diagrams may be generated, but these are the graphs responsible for renormalization. Therefore, removing divergences from the effective action one simultaneously renormalizes all diagrams to a given order. It is clear therefore that the background fields must be essentially unconstrained to allow arbitrary (small) variations. In this section, our purpose is rather moderate: we like to calculate the charge renormalization only. That is, we are interested in the coefficient in front of $F_{\mu\nu}F^{\mu\nu}$ in divergent part of the effective action. Clearly, to calculate just this coefficient it is sufficient to consider a rather restricted set of background fields (as, e.g., a stationary field satisfying classical equations of motion). For such fields we may safely use the construction of the effective action presented above in this Chapter. For generic background fields one has to use a more profound scheme based on the Legendre transform, which is explained in any textbook on quantum field theory.

QCD is a gauge theory of the color $SU(3)$ group with the spinors (quarks) belonging to the fundamental representation and the gauge fields called gluons with the classical action

$$I = \int d^4x \left[ \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi} \not{D} \psi \right], \tag{7.110}$$

with the Dirac operator $\not{D} = i\gamma^\mu D_\mu(A)$, $D_\mu(A) = (\partial_\mu + g A^a_\mu T_a)$. Here $T_a = i\lambda_a/2$ are anti-Hermitean generators of the algebra $su(3)$ corresponding to the color group $SU(3)$, and $\lambda_a$ are the Gell-Mann matrices. These generators satisfy the conditions

$$\mathrm{tr}\, T_a T_b = -\frac{1}{2}\delta_{ab}, \qquad [T_a, T_b] = f_{abc} T_c, \tag{7.111}$$

where the structure constants $f_{abc}$ are totally antisymmetric. These structure constants are used to construct the field strength $F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g f_{abc} A^b_\mu A^c_\nu$ in the usual way. The gluon fields belong to the adjoint representation of the algebra $su(3)$ where the basic generators are represented by the structure constants, $\mathrm{ad}(T_a)_{bc} = -f_{abc}$. The corresponding trace form is

$$\mathrm{tr}(\mathrm{ad}(T_a)\,\mathrm{ad}(T_b)) = f_{acd} f_{adc} = -3\delta_{ab}. \tag{7.112}$$

The color indices $a$, $b$, $c$ are moved up and down with the Kronecker delta, which makes their actual position inessential. The gluon fields are proportional to the unit matrix in flavor indices. For simplicity we suppose that quarks are light and their masses are neglected.

The quantization can be performed with the help of Exercises 2.9, 2.10 by using explicit mode analysis, as in case of QED. Here we apply the quantization scheme of Sect. 7.9. The both methods, of course, yield the same results. Let $B_\mu$ denote the background gluon field, and $A_\mu$ denote fluctuations over the background. We use a set of Lorentz-like background gauge conditions

$$0 = \mathscr{F}^c(A) = (D_\mu(B) A^\mu)^c = \partial_\mu A^{c\mu} + g f_{abc} B^a_\mu A^{b\mu}. \tag{7.113}$$

We use the same letter for the covariant derivatives acting on spinors and vector fluctuations since they are the same objects taken in two different representations. The linearized gauge transformation of $A_\mu$ reads

$$\delta_\xi A^a_\mu = D_\mu(B)\xi^a. \tag{7.114}$$

By collecting everything together, substituting in (7.109), and truncating to the quadratic order in $A_\mu$ and $\psi$ one obtains the following gauge fixed action:

$$I_{\text{gauge fixed}} = \int d^4x \left[ \frac{1}{2} A^{a\mu} (-D^2(B)^{ab} \delta_{\mu\nu} + 2g F(B)^c_{\mu\nu} f_{abc}) A^{b\nu} + \bar{\psi} \not{D}(B) \psi \right.$$
$$\left. + \bar{c}^a D^2(B)^{ab} c^b \right]. \tag{7.115}$$

Consequently, the effective action consists of two contributions

$$W[B] = W_f[B] + W_g[B]. \tag{7.116}$$

Here, $W_f[B]$ is the effective action of quarks in the background gauge field,

$$W_f[B] = -\ln\det_{\varrho} \not{D}(B). \tag{7.117}$$

The quark part $W_f[B]$ is an analog of electron-positron effective action (7.117) in QED. Since the gluons have a self-interaction they contribute to the total action as well:

$$W_g[B] = \frac{1}{2}\ln\det_{\varrho^2}\left(-\delta^\mu_\nu D^2(B) + 2gF^\mu{}_\nu\right) - \ln\det_{\varrho^2}\left(-\mathscr{D}^2(B)\right), \tag{7.118}$$

where we suppressed the color index structure.

One can now proceed as in Sect. 7.7 to get (compare with (7.95))

$$W[B] = C - \frac{1}{2}\left(a_4^{(1)} - 2a_4^{(0)} - a_4^{(1/2)}\right)\ln l^{-2}\delta + \cdots, \tag{7.119}$$

where the dots denote non-divergent terms of higher order in the gauge strength or in its derivatives. Since the fields are massless, an infrared cutoff parameter $l$ associated with a size of the system has been introduced. The value of $l$ plays no role for further analysis. The heat kernel coefficients $a_4^{(1)}$, $a_4^{(0)}$, $a_4^{(1/2)}$ correspond to the operators

$$L^{(1)} = -\delta^\mu_\nu D^2(B)^{ab} + 2gF^\mu_{c\nu} f_{abc}, \tag{7.120}$$

$$L^{(0)} = -D^2(B)^{ab}, \tag{7.121}$$

$$L^{(1/2)} = \not{D}^2(B) = -D^\mu(B)D_\mu(B) + \frac{g}{4}[\gamma^\mu, \gamma^\nu]F^a_{\mu\nu}T_a. \tag{7.122}$$

To complete the calculation it only remains to calculate the traces in (4.58). Note that for the operator $L^{(1)}$ the trace is taken over vector and color indices, while for $L^{(0)}$—over color indices only. In both cases condition (7.112) is useful. The trace for $L^{(1/2)}$ is a spinor trace, a color trace in the fundamental representation (see normalization condition (7.111)), and a trivial trace over the flavor indices which produces a factor of $N_f$ in (7.125) below

$$a_4^{(1)} = \frac{5g^2}{16\pi^2}\int d^4x\, F_{a\,\mu\nu}F_a{}^{\mu\nu}, \tag{7.123}$$

$$a_4^{(0)} = -\frac{g^2}{64\pi^2}\int d^4x\, F_{a\,\mu\nu}F_a{}^{\mu\nu}, \tag{7.124}$$

$$a_4^{(1/2)} = \frac{g^2 N_f}{48\pi^2}\int d^4x\, F_{a\,\mu\nu}F_a{}^{\mu\nu}, \tag{7.125}$$

where $F_{a\,\mu\nu}$ is always the strength of the background field $B$. Therefore,

$$a_4^{(1)} - 2a_4^{(0)} - N_f a_4^{(1/2)} = \frac{g^2}{16\pi^2}\left(\frac{11}{2} - \frac{N_f}{3}\right)\int d^4x\, F_{a\,\mu\nu}F_a{}^{\mu\nu}. \tag{7.126}$$

The total action is the sum of the classical Yang-Mills action and a quantum correction,

$$\Gamma[B] = \frac{1}{4}\int d^4x\, F_{a\,\mu\nu} F_a{}^{\mu\nu} + W[B] = \frac{Z(\delta)}{4}\int d^4x\, F_{a\,\mu\nu} F_a{}^{\mu\nu} + \cdots, \quad (7.127)$$

$$Z(\delta) = 1 - \frac{g^2}{8\pi^2}\left(\frac{11}{2} - \frac{N_f}{3}\right)\ln l^{-2}\delta, \quad (7.128)$$

where we used (7.119). The renormalization is reduced to the redefinition of fields and the coupling, $\bar{B}_\mu = \sqrt{Z} B_\mu$, $\bar{g} = g/\sqrt{Z}$. The non-Abelian strength tensor $F_{a\,\mu\nu}$ in the classical Yang-Mills action includes the term $g f_{abc} B^b_\mu B^c_\nu$. This term is renormalized under the above redefinitions. The relation between the physical coupling constant, $\bar{\alpha}_g = \bar{g}^2/(4\pi)$, and the bare coupling, $\alpha_g = g^2/(4\pi)$, follows from (7.128)

$$\frac{1}{\bar{\alpha}_g} = \frac{1}{\alpha_g(\delta)} = \frac{1}{\alpha_g} - \frac{1}{2\pi}\left(\frac{11}{2} - \frac{N_f}{3}\right)\ln l^{-2}\delta. \quad (7.129)$$

The running coupling $\alpha_g(\mu)$ is defined by (7.129) with some energy scale $\mu = 1/\sqrt{\delta}$. This yields the famous QCD beta-function

$$\beta(\alpha_g) = \mu^2 \frac{d\alpha_g(\mu^2)}{d\mu^2} = -\frac{\alpha_g^2}{6\pi}\left(\frac{33}{2} - N_f\right). \quad (7.130)$$

The sign of the beta-function depends on number of quark flavors $N_f$. The sign is negative if $N_f \leq 16$, which is the case of QCD. Thus the QCD running coupling decreases at short distances and quark interactions become weaker. This is an important physical phenomenon known as the asymptotic freedom.

## 7.10 Literature Remarks

Quantum field theories at finite temperatures have been formulated in pioneering works by E. Fradkin, T. Matsubara, J. Schwinger, see [183, 184] in the middle of the twentieth century. They have a number of important applications ranging from QCD to physics of the early universe. It was not the aim of this Chapter to give a consistent introduction to finite temperature theories. Rather we used this formalism because the corresponding wave operators are of Laplace type and spectral functions are well-defined. More details can be found in vast existing literature. We mention just few references. Functional integral representation of the partition function is introduced in the classical book by R. Feynman and A. Hibbs [109]. A review of finite temperature theories can be found in [119, 177], a fundamental modern monograph is [166].

The complex manifolds and a Wick rotation in gravity theories were first motivated and described in works by Hartle and Hawking [154] and by Gibbons and Hawking [129–131]. The metric on complex manifolds $\mathscr{M}_E$ has the positive-definite signature but contains complex components, see Sect. 7.4. If $\mathscr{M}$ is a stationary manifold which describes a solution to the Einstein equations the metric of its complex counterpart, $\mathscr{M}_E$, can be made real if some of its parameters are analytically continued to imaginary values. After that $\mathscr{M}_E$ becomes a genuine Riemannian manifold. The obtained geometries are solutions to the Euclidean Einstein equations and are called the gravitational instantons. An example of a gravitation instanton is given in Sect. 1.4. The Euclidean approach can be used to formulate thermodynamics of black hole. One can show [130] that Euclidean Einstein action on a black hole instanton is analogous to a free energy and can be used to infer thermodynamical characteristics of a black hole. Properties of gravitational instantons are discussed in [99].

The effective action is an integral part of almost all modern text books on quantum field theories, see e.g. [161, 163, 205, 253]. So does a discussion of the asymptotic freedom discovered by Gross and Wilczek [145], and Politzer [209]. The authors of this discovery were awarded the Nobel Prize in Physics in 2004. A pedagogical introduction to quantization of gauge theories and Faddev-Popov ghosts [107] is presented in [108]. The methods of renormalization group theory are presented in full in [241].

The number of references where effective action in external backgrounds and its applications is discussed is too big to be reviewed. The classical book here is by Birrell and Davies [37]. Some other monographs are [18, 53, 100, 101, 103–105, 193, 194]. Few more known research papers useful for introduction in the subject are: a classical paper on finite temperature quantum theory in static space-times is [93], a pioneering calculation of the effective action on $S^4$ with application to phase transitions in de Sitter universe is [4]. More properties of the effective action is discussed in the exercises below and in the next Chapter.

Recommended Exercises are 7.4, 7.7, 7.8, 7.9, 7.12–7.14, 7.16, 7.19, 7.21.

## 7.11 Exercises

**Exercise 7.1** Let $L$ be a second order positive operator with a discrete spectrum. Find arguments in favor of the following definition of the functional integral

$$\int [D\varphi] e^{-(\varphi, L\varphi)} \equiv (\det L)^{-1/2}, \tag{7.131}$$

where $\ln\det L$ is determined by using Ray-Singer formula (5.46).

**Exercise 7.2** Let $\not{D}$ be a selfadjoint Dirac-type operator which does not have zero eigenvalues. Use Berezin rules (7.37) to define the following path integral:

$$\int [D\bar{\psi}][D\psi] e^{-(\bar{\psi}, \not{D}\psi)} \equiv (\det \not{D}). \tag{7.132}$$

Moreover, $\ln(\det \not{D}) \equiv -\frac{1}{2}\zeta'(0, \not{D}^2)$ if the spectrum of $\not{D}$ is symmetric.

**Exercise 7.3** Show that the thermodynamical entropy can be represented as

$$S \simeq \ln N, \tag{7.133}$$

where $N$ is the number of microscopic states corresponding to given macroscopic parameters (energy, pressure and etc.).

**Exercise 7.4** Find asymptotics of the free energy of a quantum scalar field in a thermodynamical limit. Suppose that system is in a cubic box of a volume $V = l^{n-1}$ in a $n$-dimensional Minkowski space-time and the temperature $T$ is high, $Tl \gg 1$. Use this formula to reproduce the Stefan-Boltzmann law for the energy of black body radiation in $n = 4$

$$\mathscr{E} = \frac{\pi^2}{45} V T^4. \tag{7.134}$$

**Exercise 7.5** Calculate the free-energy of a massless scalar field on an interval of the length $l$ (in two-dimensional Minkowski space-time). Consider the thermodynamical limit $Tl \gg 1$ and find subleading corrections to the leading term in the free energy discussed in Exercise 7.4.

**Exercise 7.6** Define the free energy of a massless scalar field on a circle of length $l$ (in two-dimensional Minkowski space-time) which rotates with a constant angular velocity $\Omega$ ($\Omega < 2\pi/l$). Calculate the free energy in the thermodynamical limit.

**Exercise 7.7** Use formulas [142]

$$\sum_{l=1}^{\infty} \ln\left(1 + \frac{\omega^2}{\sigma_l^2}\right) = \ln\left(1 - e^{-\beta\omega}\right) + \frac{\beta\omega}{2} - \ln \beta\omega, \tag{7.135}$$

$$\sum_{l=0}^{\infty} \ln\left(1 + \frac{\omega^2}{\tilde{\sigma}_l^2}\right) = \ln\left(1 + e^{-\beta\omega}\right) + \frac{\beta\omega}{2} - \ln 2, \tag{7.136}$$

to prove (7.26) and (7.29). The Matsubara frequencies $\sigma_l$, $\tilde{\sigma}_l$ are given by (7.25) and (7.32), respectively.

**Exercise 7.8** Consider the wave operator of the form $P_E(z) = z^2 + L(iz)$, where $L(iz)$ is a Laplace type operator and $z$ is a complex parameter having the meaning of frequency, see (7.41). Use the asymptotic expansions

$$\mathrm{Tr}_{\mathbb{L}^2} \exp(-tP_E) \sim \sum_{p=0}^{\infty} t^{\frac{p-n}{2}} a_p(P_E),$$

$$K(L(\omega); t) = \mathrm{Tr}\, e^{-tL(\omega)} \sim \sum_{p=0}^{\infty} a_p(\omega) t^{p-n/2},$$

and decompositions (6.19),

$$a_{2k}(\omega) = \sum_{m=0}^{k} a_{m,k}\omega^m, \qquad a_{2k+1}(\omega) = \sum_{m=0}^{k} b_{m,k}\omega^m,$$

to prove the following representation for the heat coefficients $a_p(P_E)$:

$$a_{2k}(P_E) = \frac{\beta}{2\pi} \sum_{m=k}^{2k} (-1)^{k-m} \Gamma\left(m-k+\frac{1}{2}\right) a_{2(m-k),m}, \tag{7.137}$$

$$a_{2k+1}(P_E) = \frac{\beta}{2\pi} \sum_{m=k}^{2k} (-1)^{k-m} \Gamma\left(m-k+\frac{1}{2}\right) b_{2(m-k),m}. \tag{7.138}$$

**Exercise 7.9** Use (7.137) and results of Chap. 6 to prove the following relation between the heat coefficient $a_n(P_E)$ of the operator $P_E(z) = z^2 + L(iz)$ and the coefficient $a_n$ in pseudo-trace expansion (6.17) for a NLSP associated with the operator $L(\omega)$:

$$a_n(P_E) = \frac{\beta}{\sqrt{4\pi}} a_n. \tag{7.139}$$

The relation holds on a space-time with even dimensions $n$, corresponding NLSP (6.3) is $(n-1)$-dimensional.

**Exercise 7.10** Let $f_1(s)$ be defined by (7.52). Prove that $-f_1'(0)$ in equality (7.60) can be related with a (suitably regularized) vacuum energy

$$E_0 = \frac{1}{2} \sum_i \omega_i.$$

**Exercise 7.11** Consider representation (7.51) for the zeta-function $\zeta(s; P_E)$. Prove that the singular part of the function $f_1(s)$ (see definition (7.52)) in the complex plane of the parameter $s$ reproduces poles of $\zeta(s; P_E)$. Use formulas (7.137), (7.138).

**Exercise 7.12** Consider a free complex field $\varphi$ which obeys the Bose statistics (a scalar field, for example) and its decomposition into creation and annihilation operators

$$\varphi(x) = \sum_i a_i f_i^{(+)}(x) + \sum_j b_j^+ f_j^{(-)}(x),$$

see Sect. 2.1, Eq. (2.25). It is assumed that $\varphi$ is defined on a stationary background and the corresponding single-particle spectrum is strictly positive, $\omega_i^{(\pm)} > 0$. Find a representation of the Wightman functions

$$G_\beta^+(x,x') = \langle \hat{\varphi}(x) \hat{\varphi}^+(x') \rangle_\beta, \qquad G_\beta^-(x,x') = \langle \hat{\varphi}^+(x') \hat{\varphi}(x) \rangle_\beta$$

in terms of single particle modes. By using this representation show that $G_\beta^+$, as a function of the time variable $t$, can be analytically continued in the complex plane $z = t + i\tau$ and defined in the strip $-\beta < \tau < 0$, $-\infty < t < \infty$. Analogously, $G_\beta^-$ can be defined in the strip $0 < \tau < \beta$, $-\infty < t < \infty$.

**Exercise 7.13** On a stationary background the Green's functions depend on a single time coordinate. It is convenient to use the following notation:

$$G^{\pm}_{\beta}(t; y, y') \equiv G^{\pm}_{\beta}(x, x'),$$

where the arguments are $x = (t, y^k)$, $x' = (0, (y')^k)$ and $y^k$ are purely spatial coordinates. Let us define a new Green's function, $\tilde{G}_{\beta}(z, y, y')$, such that

$$\tilde{G}_{\beta}(z, y, y') = G^{+}_{\beta}(z, y, y'), \quad \text{if } \Im z < 0, \tag{7.140}$$

$$\tilde{G}_{\beta}(z, y, y') = G^{-}_{\beta}(z, y, y'), \quad \text{if } \Im z > 0, \tag{7.141}$$

where $z = t + i\tau$. By using results of the Exercises 2.4 and 7.12 prove the following properties (which hold in case of fields with the Bose statistics):

i) $\tilde{G}_{\beta}(z, y, y')$ is an analytic function of $z$ everywhere in the strip $-\beta < \mathrm{Im}\, z < \beta$, $-\infty < t < \infty$ except the domains where the Wightman functions $G^{\pm}_{\beta}$ have singularities;
ii) there is a periodicity property

$$\tilde{G}_{\beta}(z - i\beta, y, y') = \tilde{G}_{\beta}(z, y, y') \tag{7.142}$$

which allows one to continue $\tilde{G}_{\beta}$ further in the complex plane.

**Exercise 7.14** Introduce the function $G(\tau, y, y') \equiv \tilde{G}_{\beta}(-i\tau, y, y')$. Based on results of Exercise 7.13 prove the following properties:

i) $G$ can be written as a time-ordered correlator,

$$\begin{aligned} G(\tau, y, y') &= \langle T_E(\varphi_E(x)\varphi^{+}_{E}(x'))\rangle_{\beta} \\ &= \theta(\tau)\langle(\varphi_E(x)\varphi^{+}_{E}(x'))\rangle_{\beta} + \theta(-\tau)\langle(\varphi^{+}_{E}(x')\varphi_E(x))\rangle_{\beta}, \end{aligned} \tag{7.143}$$

where the operators $\varphi_E$ are defined as $\varphi_E(\tau, y) \equiv \varphi(-i\tau, y)$;
ii) $G$ obeys the periodicity condition, $G(\tau + \beta, y, y') = G(\tau, y, y')$;
iii) $G$ is a solution to the equation:

$$P_E(\partial_\tau, \partial_i)G(x, x') = \delta^{(n)}(x, x'). \tag{7.144}$$

Here $\delta^{(n)}(x, x') = \delta^{(n)}(x - x')/\sqrt{|\det(g_E)_{\mu\nu}|}$ and components of the metric are defined in (7.64).

**Exercise 7.15** Prove that $\langle\varphi\rangle_{\beta} = 0$ if $\varphi$ is a free field.

**Exercise 7.16** Consider a scalar field theory with the classical potential

$$U(\varphi) = \frac{\lambda}{4}(\varphi^2 - \mu^2)^2. \tag{7.145}$$

It is supposed that $\lambda > 0$. To fix the parameters of the corresponding Coleman-Weinberg potential in a quantum theory the following conditions can be imposed:

$$\Omega'(\varphi = \pm\mu) = 0, \qquad \Omega''(\varphi = \pm\mu) = 2\lambda\mu^2. \tag{7.146}$$

Give an interpretation of these conditions and derive the Coleman-Weinberg potential (7.90) for this model by using (7.146).

**Exercise 7.17** Consider a model which consists of a charged complex scalar field interacting with an Abelian gauge field. The classical Lagrangian is

$$L[\varphi, A_\mu] = -\frac{1}{2}(D_\mu\varphi)^* D_\mu\varphi - \frac{\lambda}{4}(|\varphi|^2 - \mu^2)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{7.147}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu - ieA_\mu$. The theory is invariant with respect to local $U(1)$ gauge transformations. The classical potential has a continuous family of minima at $|\varphi| = \mu$. Model (7.147) can be used to illustrate the Higgs mechanism; the gauge group is spontaneously broken in the vacuum state because the gauge field acquires a mass $m_v^2 = e^2\mu^2$ when $|\varphi| = \mu$.

Calculate the Coleman-Weinberg potential for model (7.147) in the regime when $e^2 \gg \lambda$. Show that in the ground state quantum corrections result in appearance of a new minimum where the symmetry is restored.

**Exercise 7.18** Coleman-Weinberg potential (7.90) was derived by assuming that $U''(\phi) > 0$. If $U''(\phi) < 0$ the single-particle spectrum contains modes with imaginary frequencies. These are the modes whose spatial momenta $|\mathbf{p}|$ are smaller than the "mass" $m = \sqrt{-U''(\phi)}$. Although such modes do not contribute to energy (see the discussion in the end of Sect. 6.2) and they should not be quantized, one can formally use the following definition of the ground state effective potential

$$\Omega(\phi) = U(\phi) + V^{-1}\sum_i \omega_i, \tag{7.148}$$

where summation goes over both real and imaginary energies with a convention that $\Re\,\omega_i > 0$ and $\Im\,\omega_i < 0$.

Use (7.148) to extend (7.90) to the case $U''(\varphi) < 0$ and derive the following formula:

$$\begin{aligned}\Omega(\phi) &= U(\phi) + \frac{1}{64\pi^2}\big(U''(\phi^2)\big)^2\big[\ln|U''(\phi^2)| - i\pi\theta\big(|U''(-\phi^2)\big)\big] \\ &= U(\phi) + \frac{1}{64\pi^2}\big(U''(\phi^2)\big)^2\ln\big(U''(\phi^2) - i\varepsilon\big).\end{aligned} \tag{7.149}$$

The presence of complex energy modes results in an instability of the system. Show that the imaginary part, $\Im\,\Omega(\phi)$, can be used to find a probability for the system to decay in a unit volume per unit time.

**Exercise 7.19** The high temperature limit is one of the regimes when the effective action can be computed analytically and has a number of important applications.

Consider a theory whose single particle spectrum in a stationary spacetime is described by a non-linear spectral problem which can be brought to form (6.3). Suppose that the coefficients $a_p$ of asymptotic expansion (6.17),

$$K(t) = \frac{1}{2}\sum_\omega e^{-t\omega^2} \sim \sum_{p=0}^{\infty} a_p t^{p-(n-1)/2},$$

are known. Show that at large temperatures the effective action of the theory (7.2) can be approximated as

$$W[\phi_E] \simeq -\frac{2^n}{\sqrt{\pi}\beta^{n-1}} \sum_{p=0}^{n-1} \gamma_{n,p} \Gamma\left(\frac{n-2p}{2}\right) \zeta(n-2p) a_{2p} \left(\frac{\beta}{2}\right)^{2p} + \frac{\beta}{\sqrt{\pi}} a_n \ln(\mu\beta), \tag{7.150}$$

where $\zeta(x)$ is the Riemann zeta-function (5.5). The expansion is applicable to the case of a complex scalar field or a Dirac field on a closed manifold of the space-time dimension $n$. The coefficient $\gamma_{n,p} = 1$ for Bosons and $\gamma_{n,p} = 1 - 2^{2p+1-n}$ for Fermions.

The last term in the r.h.s. of (7.150) is absent for $n$ odd. The parameter $\mu$ is proportional to the mass gap of the theory.

**Exercise 7.20** Use (7.150) to calculate the effective potential for the model (7.145) at high temperatures and show that the mean field in this state vanishes.

**Exercise 7.21** Consider an electron-positron plasma at a high temperature $T$ in an external electric field. Compute the free energy (or the effective action) of the plasma by using approximation (7.150) and show that the electric field of a test charge $e$ is screened at distances $1/M(T)$, where $M(T) = \sqrt{\frac{1}{3}e^2T^2}$ is the so-called Debye mass.

**Exercise 7.22** Consider a formal decomposition of effective action (7.92) in QED

$$W[A] = \sum_{k=0}^{\infty} \int d^4x_1\, d^4x_2 \dots d^4x_{2k}\, W_{(2k)}^{\mu_1,\mu_2,\dots,\mu_{2k}}(x_1, x_2, \dots, x_{2k}) A_{\mu_1}(x_1) \dots A_{\mu_{2k}}(x_{2k}). \tag{7.151}$$

Prove that the coefficients $W^{(2k)}$ are given by formula (7.97).

# Chapter 8
# Quantum Anomalies

## 8.1 Noether Theorems

The aim of this Chapter is to use the spectral methods to demonstrate violation of classical symmetries in quantum models. The symmetry violation appears as non-conservation on a quantum level of certain quantities which are 'integrals of motion' in the classical theory. One calls this property quantum anomalies.

A general relation between classical symmetries and conservation laws in physics is given by the first and second Noether theorems. The first theorem applies when a system has a finite-dimensional continuous Lie group of symmetries, for example an axial symmetry or translational invariance. The theorem states that there exist quantities, so-called Noether charges, which are conserved in time, and the number of the charges equals the dimensionality of the symmetry group. The second Noether theorem is applied when there is an infinite-dimensional continuous group of symmetries and it yields covariant conservation laws for certain currents. Examples of such symmetries are diffeomorphisms and local gauge transformations.

To give an idea of the Noether theorems, consider a classical action $I[\varphi,\phi]$ with the dynamical variables $\varphi$ and background fields $\phi$. Suppose that transformations of an infinite-dimensional continuous group $G$ do not change the action, $I[\varphi,\phi]=I[\varphi',\phi']$. By using invariance of the action and boundary conditions one arrives at the following identity:

$$\delta_\lambda I[\varphi,\phi]=\int d^n x\left[\frac{\delta I[\varphi,\phi]}{\delta\phi}\delta_\lambda\phi+\frac{\delta I[\varphi,\phi]}{\delta\varphi}\delta_\lambda\varphi\right]=0, \tag{8.1}$$

where $\delta_\lambda\phi=\phi'-\phi$, $\delta_\lambda\varphi=\varphi'-\varphi$ are infinitesimal variations of the variables generated by the group $G$ with a set of parameters $\lambda$. The variations with respect to the dynamical variables $\varphi$ vanish if the fields satisfy the equations of motion. To proceed with the second term in the r.h.s. of (8.1) one should specify the transformations of the background fields. Quite generically, one can write

$$\delta_\lambda\phi^a(x)=(d^{ab\mu}\nabla_\mu+f^{ab}{}_c\phi^c(x))\lambda_b(x). \tag{8.2}$$

Here $a,b,c$ are sets of indexes carried by the field, $d^{ab\mu}$, $f^{ab}{}_c$ are some matrices, the form of these matrices being specified by the symmetry. The group parameters

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$\lambda^b(x)$ are some arbitrary sufficiently smooth functions which are assumed to have compact supports. Equations (8.1), (8.2) then imply a "conservation law"

$$(d^{ab\mu}\nabla_\mu - f^{ab}_{\ \ c}\phi^c(x))J_a(x) = 0, \tag{8.3}$$

$$J_a(x) = g^{-1/2}\frac{\delta I}{\delta\phi^a(x)}, \tag{8.4}$$

where $g$ is the determinant of the background metric. We shall see that Eqs. (8.3) have a covariant form.

The quantities $J_a$ are called the Noether currents. Equations (8.3), (8.4) express the second Noether theorem and, in accord with the theorem, the Noether currents are "conserved" provided the dynamical variables $\varphi$ obey classical equations of motion.

If $\varphi$ are quantized on a classical background $\phi$, one defines the effective action $W[\phi]$. This functional is an analog of classical functional $I[\varphi,\phi]$ discussed in Sect. 7.1. In the quantum theory the classical currents $J_a$ are replaced with quantum averages of the corresponding operators,

$$\langle J_a(x)\rangle = g^{-1/2}\frac{\delta W[\phi]}{\delta\phi^a(x)}, \tag{8.5}$$

in accord with (7.7). We shall see below that in certain cases the effective action $W[\phi]$ may not be invariant under transformations of $\phi$ generated by the continuous group $G$ even if $G$ is a symmetry group of the corresponding classical action $I[\varphi,\phi]$. If the variation of $W[\phi]$ is non-vanishing,

$$\delta_\lambda W = \int d^n x\,\sqrt{g}\mathscr{A}_a(x)\lambda^a(x) \neq 0, \tag{8.6}$$

the Noether identity (8.3) for the quantum currents is violated and takes the form

$$(d^{ab\mu}\nabla_\mu - f^{ab}_{\ \ c}\phi^c(x))\langle J_a(x)\rangle = -\mathscr{A}_a(x). \tag{8.7}$$

The right hand side of (8.7) is called the quantum anomaly. We consider now some typical examples.

**Gauge Symmetries** The first example is related to models of scalar and spinor fields interacting with a background vector potential $A_\mu$, see Eqs. (1.68), (1.73), respectively. The models are invariant with respect to transformations of a local gauge $U(1)$ group. Transformation of the background field $\delta_\lambda A_\mu = -\nabla_\mu\lambda$ is a simplest form of (8.2). The Noether current calculated with the help of (8.4) coincides with usual electric current (1.71).

If spinors are massless and space-time is even dimensional the theory also allows a global $U(1)$ group with transformations $\psi' = e^{-i\gamma_\star\lambda}\psi$. The parts of a spinor with different chiralities (introduced in Sects. 5.9 and 5.10) then transform in different ways. This symmetry is called chiral. The chiral transformations can be made local and become a gauge symmetry if an additional gauge potential is introduced to compensate terms in the action which appear due to transformations of spinors.

One can then define a corresponding conserved Noether current called the axial current. In a quantum theory the chiral symmetry is broken and the axial current is not conserved. This effect which is studied in details in Sect. 8.2 is called the axial anomaly.

**Diffeomorphism Invariance** The equivalence principle requires that action of matter fields in the presence of gravity is invariant with respect to coordinate transformations. The diffeomorphisms generate transformations of the metric and matter fields, $\delta_\xi g_{\mu\nu} = \mathscr{L}_\xi g_{\mu\nu}$, $\delta_\xi \phi = \mathscr{L}_\xi \phi$, which have form (8.2), see Sect. 1.7 and (1.81), (1.83). The Noether currents corresponding to the group of diffeomorphisms are determined by (8.4) in terms of the variations over the background metric and coincide with components of the stress-energy tensor $T^{\mu\nu}$ defined by (1.22). It follows from the second Noether theorem that $T^{\mu\nu}$ has zero covariant divergence

$$\nabla_\mu T^{\mu\nu} = 0. \tag{8.8}$$

If the background manifold has isometries, a number of integrals of motion appear according to the first Noether theorem. If $\xi^\mu$ is a Killing field which generates the given group of isometries the corresponding charge is

$$Q(\xi) = \int_\Sigma d\Sigma^\mu \, \theta_\mu(\xi), \tag{8.9}$$

$$\theta_\mu(\xi) = T_{\mu\nu}\xi^\nu, \qquad \nabla^\mu \theta_\mu(\xi) = 0, \tag{8.10}$$

where the integral goes over a space-like hypersurface $\Sigma$. Validity of the conservation law for the current $\theta_\mu(\xi)$ and independence of $Q(\xi)$ on $\Sigma$ is guaranteed by (8.8) and Killing equation (1.82). As an example, one can mention stationary space-times where $\xi$ is time-like Killing field which generates time translations. The Noether charge (8.9) in this case is just the energy of a system.

As we shall see in Sect. 8.4, even the fundamental law (8.8) may be violated in quantum theories. This happens, e.g. in models of quantized chiral spinors. Appearance of a non-trivial term in the r.h.s. of (8.8) is known as a gravitational or Einstein anomaly.

**Conformal Symmetry** If the classical action does not contain dimensional constants like masses it may be invariant with respect to local conformal transformations of the metric, $\delta_\sigma g_{\mu\nu} = -2\sigma g_{\mu\nu}$, and corresponding rescalings of matter fields. It is not difficult to see from (8.2) that Noether condition (8.3) is reduced to

$$T^\mu_\mu = 0. \tag{8.11}$$

Models having this symmetry belong to a class of the so-called conformal field theories. Noether conditions (8.11) do not include derivatives of the currents because the derivatives do not appear in the transformation $\delta_\sigma g_{\mu\nu}$.

If the space-time admits a conformal vector field $\xi$, the first Noether theorem predicts another conserved charge $Q(\xi)$. The charge is still defined by (8.9), (8.10).

Condition (8.8) and conformal Killing equation (1.86) guarantee that the Noether current $\theta(\xi)$ has zero divergence.

In quantum case, the trace of the expectation value of the stress-energy tensor usually is not zero even in conformal theories. This phenomenon is called the Weyl anomaly or conformal anomaly. We return to it in Sect. 8.5.

**Local Lorentz Symmetry** In a theory on a gravitational background there is an additional group of invariance. The group acts on vielbeins $e_a^\mu(x)$ as local $SO(n)$ or $SO(1,n-1)$ transformations depending on whether the background is a Riemannian or a Lorentzian manifold, see (1.46) and Sect. 1.5.

To be more specific we suppose that the manifold is Riemannian. From variations of the vielbeins, $\delta_\lambda e_a^\mu = M_{ab} e_b^\mu \lambda$, and from (8.2) one then arrives at the following Noether condition:

$$T^a{}_\mu e_b^\mu M_{ab} = 0, \tag{8.12}$$

$$T^a{}_\mu \equiv -e^{-1} \frac{\delta I}{\delta e_a^\mu}, \tag{8.13}$$

where $e = \det e_\mu^a = \sqrt{g}$. In a pure metric theory one can use Eq. (1.43) for vielbeins and relate (8.13) with definition of the stress-energy tensor (1.22),

$$T^\nu{}_\mu = e_a^\nu T^a{}_\mu . \tag{8.14}$$

Thus, (8.12) can be written in the form

$$T_{\mu\nu} M^{\mu\nu} = 0, \tag{8.15}$$

where $M^{\mu\nu} = e_a^\mu e_a^\nu M_{ab}$. The generators $M_{ab}$ of $SO(n)$ group are antisymmetric matrices, $M_{ab} = -M_{ba}$. Therefore, Eq. (8.15) is satisfied identically because the metric stress-energy tensor is symmetric.

In quantum theory of spinor fields the vielbeins appear in the effective as independent variables. That is why the stress-energy tensor cannot be derived as a variation over the metric and it is not a priory symmetric. This is precisely what happens in models of quantized chiral spinors where $T^{\mu\nu} M_{\mu\nu} \neq 0$. This property is called the Lorentz anomaly and it is closely related to the gravitational anomaly. We shall return to this subject in Sect. 8.3.

We now give derivation of the above mentioned anomalies in different models by using results of Sects. 5.7 and 5.10.

## 8.2 Axial Anomaly

Our first example is a well-known violation of a chiral gauge symmetry. We consider spin 1/2 fields on an even-dimensional Riemannian manifold $\mathcal{M}$ without boundaries. The corresponding classical action is

$$I[\psi, V, A] = \int_{\mathcal{M}} d^n x \sqrt{g} \psi^+ \not{D} \psi . \tag{8.16}$$

Because the base manifold has the Euclidean signature the Hermitian conjugation $\psi^+$ is used instead of the Dirac one, compare with (1.73). Let us take the Dirac operator in the form

$$\not{D} = i\gamma^\mu(\partial_\mu + V_\mu + i\gamma_* A_\mu), \tag{8.17}$$

where $V_\mu$ and $A_\mu$ are background vector and axial vector fields, respectively. The fields are supposed to take values in a representation of a Lie algebra of some compact gauge group $G$. They are also supposed to be anti-Hermitian matrices. This guarantees that Dirac operator (8.17) is symmetric or formally self-adjoint with respect to the product

$$(\psi_1, \psi_2) = \int_{\mathscr{M}} d^n x\, \bar{\psi}_1(x)\psi_2(x), \tag{8.18}$$

see Sect. 3.1.

Consider the following two types of gauge transformations:

$$\begin{aligned} &\delta_- \psi = -\rho\psi, \qquad \delta_-\psi^+ = \psi^+\rho, \\ &\delta_- A_\mu = [A_\mu, \rho], \qquad \delta_- V_\mu = \partial_\mu\rho + [V_\mu, \rho]; \end{aligned} \tag{8.19}$$

and

$$\begin{aligned} &\delta_+ \psi = -i\lambda\gamma_*\psi, \qquad \delta_+\psi^+ = -i\psi^+\lambda\gamma_*, \\ &\delta_+ A_\mu = \partial_\mu\lambda + [V_\mu, \lambda], \qquad \delta_+ V_\mu = -[A_\mu, \lambda]. \end{aligned} \tag{8.20}$$

The parameters $\rho$ and $\lambda$ are anti-Hermitian matrices depending on coordinates and taking values in the same representation of the Lie algebra of $G$ as for the background fields.

One can check that (8.19), (8.20) generate simple variations of the Dirac operator

$$\delta_- \not{D} = \not{D}\rho - \rho\not{D}, \qquad \delta_+\not{D} = i(\lambda\gamma_*\not{D} + \not{D}\lambda\gamma_*). \tag{8.21}$$

It follows then from (8.21) that transformations (8.19), (8.20) do not change the action (8.16) and, therefore, they are symmetries of the given classical theory. $\delta_+$ transformations (8.20) are known as axial gauge symmetries. Transformations (8.19), (8.20) belong to a class of variations (8.2) considered in Sect. 8.1. So one can introduce classical axial and vector Noether currents (8.4)

$$J^\mu_A = g^{-1/2}\frac{\delta I}{\delta A_\mu}, \qquad J^\mu_V = g^{-1/2}\frac{\delta I}{\delta V_\mu}, \tag{8.22}$$

and establish the concrete form for Noether identities (8.3)

$$\nabla_\mu J^\mu_V + [V^\mu, J^\mu_V] + [A^\mu, J^\mu_A] = 0, \tag{8.23}$$

$$\nabla_\mu J^\mu_A + [V^\mu, J^\mu_A] - [A^\mu, J^\mu_V] = 0. \tag{8.24}$$

Let us check which of these symmetries is violated at the level of the corresponding effective action. In terms of the regularized determinant of the Dirac operator the effective action is $-\ln\det \not{D}(V, A)$. It is convenient to use the results of Sect. 5.10 and Exercise 5.10. Relations (5.89), (5.96) and (5.98) yield the following result:

$$\delta_- \ln\det \not{D}(V, A) = 0, \tag{8.25}$$

$$\delta_+ \ln\det \not{D}(V, A) = 2i\zeta\big(0, \not{D}^2, \gamma_\star\lambda\big). \tag{8.26}$$

We noticed here that $\rho$ and $\lambda$ are anti-Hermitian and used (8.21). Axial gauge symmetries (8.20), therefore, are violated in the quantum theory and result in the axial anomaly,

$$\nabla_\mu \langle J_A^\mu\rangle + [V^\mu, \langle J_A^\mu\rangle] - [A^\mu, \langle J_V^\mu\rangle] = -\mathscr{A}, \tag{8.27}$$

$$\mathscr{A} = -2ig^{-1/2}\frac{\delta}{\delta\lambda}\zeta\big(0, \not{D}^2, \gamma_\star\lambda\big). \tag{8.28}$$

Usual gauge transformations (8.19) are not violated at the quantum level and preserve the Noether law (8.23). An example of calculations of the axial anomaly can be found in Exercise 8.1.

Even though the axial symmetry is violated at the quantum level, the Lie algebra relations (commutators) between gauge and chiral transformations remain intact. By applying these relations to the effective action one obtains important equations among the anomalies. These equations are called the Wess-Zumino consistency conditions [255].

## 8.3 Lorentz Anomaly

In this and the next sections we study gravitational anomalies that appear in quantum models of chiral spinors in space-times of even dimensions. We are going to the spectral methods developed for elliptic operators. Therefore, transition to the Euclidean space is desirable. The structure of spin representations in Euclidean and Minkowski signature spaces is quite different. Consequently, Wick rotation of spinors is a rather nontrivial procedure. There are different inequivalent prescriptions for this procedure. For a overview we refer the reader to [240]. Here we shall only present an example of difficulties which occur when chiral spinors are Wick rotated naively. In Minkowski signature even-dimensional space, if $\gamma_*\psi = \psi$, then $\bar\psi\gamma_* = -\bar\psi$. Since $\not{D}$ is block anti-diagonal in the chiral basis,

$$\not{D} = i\gamma^\mu\nabla_\mu^{(s)} = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \tag{8.29}$$

in the Lagrangian $\bar\psi\not{D}\psi$ the fermions of a fixed chirality are coupled to the fermions of the same chirality only. Hence, it is possible to define a classical action for the fermions of just one chirality. In Euclidean space the situation is different. For a positive chirality $\psi$, $\gamma_*\psi = \psi$, the conjugated spinor satisfies $\psi^\dagger\gamma_* = \psi^\dagger$. Consequently, the classical action $\psi^\dagger\not{D}\psi$ is identically zero if $\psi$ is restricted to be of a fixed (positive or negative) chirality. This explains many difficulties with chiral fermions in Euclidean space.

Nevertheless, one can do quantum calculations with chiral spinors in Euclidean space. Roughly speaking, the recipe is: calculate all traces over the spinor indices in Minkowski space and do the Wick rotation only afterwards. Then one can apply all usual methods, like the point splitting procedure, to calculate the averages.

We shall use a different approach. We want to derive the gravitational anomalies by employing the spectral methods. However, in the case of chiral theories a straightforward introduction of the effective action is problematic because one cannot define the determinant of the chiral operator $D$, see Sect. 5.10. This problem is not crucial: for physical quantities $\mathscr{O}$ which follow from variations over background fields $\phi$ one needs to know spectral functions related to variations of $D$ rather than the effective action itself. That is why we postulate that

$$\int d^n x \, \langle \mathscr{O}(x) \rangle \delta\phi(x) \equiv -\delta_\phi \ln\det \hat{D}, \tag{8.30}$$

$$\hat{D} = \begin{pmatrix} 0 & \bar{D} \\ D & 0 \end{pmatrix}. \tag{8.31}$$

Here $\bar{D}$ is a "$D_-$" type chiral operator which is assumed to be fixed under the variations. The operator $\hat{D}$ is of the Dirac type and we use Ray-Singer formula (5.46) to define $\ln\det \hat{D} = -\zeta'(0, \hat{D}^2)$. The motivation for (8.30) is as follows. If $D$ and $\bar{D}$ were finite-dimensional matrices one could write

$$\delta_\phi \ln\det \hat{D} = \mathrm{Tr}\big[\delta_\phi D \cdot D^{-1}\big].$$

This variation has the same structure as Eq. (7.8) in Sect. 7.1. By taking into account (7.8) we define the chiral effective action $W$ as a functional whose variations coincide with variations of $\ln\det \hat{D}$,

$$\delta_\phi W \equiv -\delta_\phi \ln\det \hat{D}, \tag{8.32}$$

under fixed $\bar{D}$. The background field in the considered case is the metric. Variations of $W$ can be found by using results of Sect. 5.10. After the variation has been performed, we identify the auxiliary operator as $\bar{D} = D_-$.

For simplicity, we shall use two-dimensional models on Riemannian manifolds to demonstrate the appearance of the gravitational anomalies. The corresponding structure group of a spin bundle is $Spin(2)$. It is equivalent to a group of $U(1)$ chiral gauge transformations. Let us consider this property of two-dimensional theories more carefully. It is convenient to use a complex basis of vielbeins, $e = (e^1 + ie^2)/\sqrt{2}$,

$$(e \cdot e) = (\bar{e} \cdot \bar{e}) = 0, \qquad (e \cdot \bar{e}) = 1, \tag{8.33}$$

where $\bar{e} = e^*$. The Levi-Civita connection (1.51) and the scalar curvature are

$$(w_\mu)^{ab} = -(\nabla_\mu e^a \cdot e^b) = -\bar{\varepsilon}^{ab} v_\mu, \tag{8.34}$$

$$R = \varepsilon^{\mu\nu}(v_{\mu,\nu} - v_{\nu,\mu}). \tag{8.35}$$

Here $\bar{\varepsilon}_{12} = -\bar{\varepsilon}_{12} = 1$, $\bar{\varepsilon}_{11} = \bar{\varepsilon}_{22} = 0$, $\varepsilon_{\mu\nu} = \sqrt{g}\,\bar{\varepsilon}_{\mu\nu}$, and

$$v_\mu = -\frac{1}{2}\bar{\varepsilon}_{ab}(w_\mu)^{ab} = i(\bar{e}\cdot\nabla_\mu e). \tag{8.36}$$

A Lorentz rotation of the vielbeins,

$$(e')^1 = \cos\lambda e^1 + \sin\lambda e^2, \qquad (e')^2 = -\sin\lambda e^1 + \cos\lambda e^2, \tag{8.37}$$

$$e'(x) = e^{-i\lambda(x)}e(x), \tag{8.38}$$

generates a gauge-like transformation

$$v'_\mu = v_\mu + \partial_\mu\lambda. \tag{8.39}$$

In fact, $v_\mu$ looks as a gauge potential in a sort of 2D chiral gauge theory, see Eqs. (8.16), (8.17). This interpretation follows from the form of the spin connection

$$\nabla^{(s)}_\mu\psi = \left(\partial_\mu + \frac{i}{2}\gamma_\star v_\mu\right)\psi, \tag{8.40}$$

see definition (1.57). We used in (8.40) the matrices $\gamma_\mu = e^a_\mu\sigma_a$, $a = 1, 2$, where $\sigma_a$ are the Pauli matrices, $\gamma_\star = i\sigma_1\sigma_2$. From (8.39) and (8.40) it is easy to see that Lorentz rotations of a spinor field are chiral $U(1)$ transformations

$$\psi'(x) = e^{-i\frac{\lambda(x)}{2}\gamma_\star}\psi(x). \tag{8.41}$$

One can now determine the chiral parts $D_\pm$ of operator (8.29),

$$D_+ = i\sqrt{2}\bar{e}^\mu\left(\partial_\mu + \frac{i}{2}v_\mu\right), \qquad D_- = i\sqrt{2}e^\mu\left(\partial_\mu - \frac{i}{2}v_\mu\right). \tag{8.42}$$

These definitions is our starting point for studying the effects of gravitational anomalies.

Variation of the operator $D$ under Lorentz rotations (8.41) is

$$\delta_\lambda D(e) = D'(e') - D(e) = \frac{i}{2}(\lambda D + D\lambda). \tag{8.43}$$

It follows from the results of Sect. 5.10 that the rotations change the phase $\Phi(D)$ and do not change the absolute value of the determinant. For transformations (8.43), when $\bar{D}$ at the last step is identified with $D_-$, Eq. (5.97) yields

$$\delta_\lambda\Phi(D) = \frac{1}{2}\zeta\big(0, \not{D}^2, \lambda\big). \tag{8.44}$$

One can use the relation between the zeta-function and the heat coefficients,

$$\zeta(0, L, \mathcal{O}) = a_n(\mathcal{O}, L) - \mathrm{Pr}_N(\mathcal{O}), \tag{8.45}$$

where $\mathrm{Pr}_N(\mathcal{O})$ denotes the trace of projection of the operator $\mathcal{O}$ on the space of the zero modes of operator $L$, see Eq. (5.65). With the help of (4.57), (8.32), and (8.45) one easily finds the variation of the effective action

$$\delta_\lambda W = -i\delta_\lambda \Phi(D) = 2ic \int \sqrt{g}\, d^2x\, R(x)\lambda(x) + \frac{i}{2}\mathrm{Pr}_N(\lambda), \tag{8.46}$$

where $c = 1/(48\pi)$. For simplicity, we have assumed that the parameter $\lambda(x)$ has a compact support, thus possible boundary terms do not appear in (8.46). To see how it violates the Noether condition (8.15), we take (8.13) as the definition of the stress-energy tensor and rewrite (8.46),

$$\delta_\lambda W = -\int e\, d^2x\, T^a{}_\mu \delta_L e^\mu_a. \tag{8.47}$$

As in (8.37) the Lorentz variations of vielbeins are $\delta_\lambda e^\mu_a = M_{ab} e^\mu_b \lambda$, where $M_{ab} = \bar{\varepsilon}_{ab}$. It follows then from (8.46) and (8.47) that

$$\varepsilon^{\mu\nu} T_{\mu\nu} = -2icR - \frac{iz}{2}, \tag{8.48}$$

$$z = g^{-1/2} \frac{\delta}{\delta\lambda} \mathrm{Pr}_N(\lambda) = \sum_{j=1}^{N} \psi_j^\dagger(x)\psi_j(x), \tag{8.49}$$

where $T_{\mu\nu} = g_{\mu\rho} e^\rho_a T^a{}_\nu$, see (8.14), and $\varepsilon^{\mu\nu} = e^\mu_a e^\nu_b \bar{\varepsilon}^{ab}$. $\psi_j$, $j = 1, \ldots, N$ are the zero modes of $\not{D}^2$.

The r.h.s. of Eq. (8.48) is called the Lorentz anomaly. The Lorentz anomaly consists of the two terms: a universal term determined by the curvature scalar $R$ and the contribution of zero modes, $z$. The $z$-term depends on boundary conditions and global properties of the background manifold $\mathscr{M}$. It may be absent or be unimportant because it is inverse proportional to the volume of $\mathscr{M}$.

The Lorentz anomaly makes the stress-energy tensor $T_{\mu\nu}$ non-symmetric. Therefore, such tensor cannot appear in a diffeomorphism invariant gravity theory as a variation of the action over the metric.

## 8.4 Einstein Anomaly

Consider now transformation (8.32) under a change of coordinates $(x')^\mu = f^\mu(x)$. We are interested in the variation of the operator

$$\delta_\xi D(x)\varphi(x) \equiv D'(x)\varphi(x) - D(x)\varphi(x) \tag{8.50}$$

under coordinate transformations $(x')^\mu = x^\mu - \xi^\mu(x)$ generated by a vector field $\xi^\mu(x)$. The operator $D$ acts on functions $\varphi$ which carry no indices and the objects like $D\varphi$ behave as scalars. If one takes into account that $D(x')\varphi(x') = D(x)\varphi(x)$ the r.h.s. of (8.50) can be rewritten as

$$\delta_\xi D(x)\varphi(x) = [\partial_\xi, D]\varphi(x), \tag{8.51}$$

where $\partial_\xi = \xi^\mu \partial_\mu$ and terms $O(\xi^2)$ were neglected. Variation (8.51) belongs to a class of operator transformations (5.89) discussed in Sect. 5.10. With the help of (5.93) one gets the variation of the effective action

$$\delta_\xi W = \zeta\big(0, \not{D}^2, \gamma_\star \partial_\xi\big) = a_2\big(\gamma_\star \partial_\xi, \not{D}^2\big) - \mathrm{Pr}_N(\gamma_\star \partial_\xi). \tag{8.52}$$

If the diffeomorphism generating field $\xi^\mu$ has a compact support, a straightforward computation yields,

$$\delta_\xi W = ic \int \sqrt{g}\, d^2x\, R(3v^\mu \xi_\mu + \varepsilon^{\mu\nu}\xi_{\mu;\nu}) - \mathrm{Pr}_N(\gamma_\star \partial_\xi). \tag{8.53}$$

As in the case of Lorentz transformations, (8.46), the first term in (8.53) is pure imaginary.

The computation of $a_2(\gamma_\star \partial_\xi, \not{D}^2)$ in (8.52) is based on methods of [51]. By analogy with (4.43),

$$a_p(\mathscr{O}, L) = i\partial_\varepsilon a_{p+2}(L(\varepsilon))|_{\varepsilon=0}, \tag{8.54}$$

$$L(\varepsilon) = L + i\varepsilon \mathscr{O}, \tag{8.55}$$

where $L$ is a Laplace type operator and $\mathscr{O}$ is a first order differential operator. The proof of (8.54) is rather straightforward and we leave it for the reader. In the case considered here

$$L(\varepsilon) = \not{D}^2 + i\varepsilon\gamma_\star \partial_\xi. \tag{8.56}$$

The operator $L(\varepsilon)$ can be represented as

$$L(\varepsilon) = -D_\mu(\varepsilon)D^\mu(\varepsilon) - \frac{i\varepsilon}{2}\gamma_\star \nabla\xi + \frac{\varepsilon}{2}v^\mu \xi_\mu + \frac{1}{4}R + O(\varepsilon^2), \tag{8.57}$$

$$D_\mu(\varepsilon) = \nabla_\mu^{(s)} - \frac{i}{2}\gamma_\star \varepsilon \xi_\mu \equiv \nabla_\mu + \tilde{w}_\mu. \tag{8.58}$$

The terms $O(\varepsilon^2)$ are not important for (8.54). Operator (8.57) has form (3.2) with

$$\Omega_{\mu\nu} = \frac{i}{2}\gamma_\star\big(\nabla_\mu v_\nu - \nabla_\nu v_\mu - \varepsilon(\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu)\big), \tag{8.59}$$

$$E = -\frac{i\varepsilon}{2}\gamma_\star \nabla\xi + \frac{\varepsilon}{2}v^\mu \xi_\mu + \frac{1}{4}R + O(\varepsilon^2), \tag{8.60}$$

where we used (8.57) and (1.41). Equation (4.58) for $n = 2$ yields

$$\partial_\varepsilon a_{p+2}(L(\varepsilon))|_{\varepsilon=0} = \frac{1}{48\pi}\int \sqrt{g}\, d^2x\, R\big(3v^\mu \xi_\mu + \varepsilon^{\mu\nu}\xi_{\mu;\nu}\big). \tag{8.61}$$

Together with (8.54) this relation results in (8.53).

To find the Noether conditions one should write $\delta_\xi W$ in terms of stress-energy tensor (8.13),

$$\delta_\xi W = -\int \sqrt{g}\, d^2x\, T^a{}_\nu \mathscr{L}_\xi e^\nu_a = -\int \sqrt{g}\, d^2x\, \xi_\nu\big(\nabla_\mu T^{\mu\nu} + \varepsilon_{\lambda\rho}T^{\lambda\rho}v^\nu\big). \tag{8.62}$$

Variations of the vielbeins are determined by Lie derivative $\mathscr{L}_\xi e^\nu_a = \nabla_\xi e^\nu_a - \nabla_{e_a}\xi^\nu$, see (1.83). To get (8.62) we have integrated by parts, used (8.14) and (8.36). Equations (8.53), (8.62) give

$$\nabla_\mu T^{\mu\nu} + \varepsilon_{\lambda\rho}T^{\lambda\rho}v^\nu = -ic\big(3v^\nu R + \nabla_\mu(\varepsilon^{\mu\nu}R)\big) + q^\nu, \tag{8.63}$$

$$q^\nu = g^{-1/2}\frac{\delta}{\delta\xi_\nu}\mathrm{Pr}_N(\gamma_\star\partial_\xi), \tag{8.64}$$

and, if the Lorentz anomaly (8.48) is taken into account, one gets a Noether condition associated with the coordinate transformations

$$\nabla_\mu T^{\mu\nu} = -ic\big(v^\nu R + \nabla_\mu(\varepsilon^{\mu\nu}R)\big) + z^\nu. \tag{8.65}$$

The right side of this equation is called the Einstein anomaly. The term $z^\nu$ is related to the presence of zero modes. It can be written in the following covariant form:

$$z^\nu = q^\nu + \frac{i}{2}zv^\nu = g^{-1/2}\frac{\delta}{\delta\xi_\nu}\mathrm{Pr}_N\big[\gamma_\star(\xi\cdot\nabla^{(s)})\big] = \sum_{j=1}^{N}\psi_j^\dagger\gamma_\star\nabla^{(s)\nu}\psi, \tag{8.66}$$

where we have used (8.49), (8.64) and definition (8.40).

One can add to $T^{\mu\nu}$ a number of local terms known as the Bardeen-Zumino polynomial and introduce the tensor

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + ic\big(v^{\nu;\mu} + v^{\mu;\nu} - 2g^{\mu\nu}(\nabla v)\big) + ic\varepsilon^{\mu\nu}R. \tag{8.67}$$

The modified stress-energy tensor is symmetric, $\varepsilon_{\mu\nu}\tilde{T}^{\mu\nu} = 0$, and free of the Lorentz anomaly. This transformation also brings the Noether condition (8.65) to a 'standard' form

$$\nabla_\mu\tilde{T}^{\mu\nu} = -\frac{ic}{2}\nabla_\mu(\varepsilon^{\mu\nu}R) + z^\nu \tag{8.68}$$

which is usually used in the literature (though customary the contribution of zero modes $z^\mu$ is omitted). Equation (8.68) is instructive because it shows that the Lorentz and Einstein anomalies are interrelated, and that they are different manifestations of a single phenomenon. One can add to $\tilde{T}^{\mu\nu}$ the local term $(ic/2)\varepsilon^{\mu\nu}R$ which eliminates the Einstein anomaly but leaves the modified stress-energy tensor non-symmetric.

Some remarks about the terminology used in the literature are in order. Equation (8.65) is called consistent anomaly. The word 'consistent' means that the corresponding stress-energy tensor $T^{\mu\nu}$ can be derived from an effective action in the course of a variation procedure. This is not a completely obvious statement since we defined only the variations of the effective action rather than the effective action itself. This consistent anomaly condition does not have a covariant form due to the presence of the connection term. On the other hand, the anomaly written in the form (8.68) is manifestly covariant, and it is called the covariant anomaly. The covariant anomaly is not consistent in a sense that $\tilde{T}^{\mu\nu}$ may not follow from an effective action as a result of variation over the metric.

## 8.5 Conformal Anomaly

One more example of symmetries which may be broken by quantum effects are conformal transformations of the metric, see Sect. 1.7. An example of a theory with the conformal symmetry is a massless scalar field $\varphi$ with a special type of non-minimal coupling to the curvature. The action of the theory is

$$I[\varphi, g] = -\int d^n x \sqrt{g}\left(g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi + \frac{n-2}{4(n-1)}\varphi^2 R\right), \tag{8.69}$$

where $R$ is the scalar curvature for the background metric $g_{\mu\nu}$. The functional (8.69) does not change under the following transformations:

$$\bar{g}_{\mu\nu}(x) = e^{-2\sigma(x)} g_{\mu\nu}(x), \tag{8.70}$$

$$\bar{\varphi}(x) = e^{(n-2)\sigma(x)/2}\varphi(x), \tag{8.71}$$

where $\sigma(x)$ is a sufficiently smooth function. The Noether theorem states that the stress-energy tensor of the field $\varphi$ has a vanishing trace, see (8.11).

Field theory models which are conformally invariant on the classical level usually loose this property after the quantization. This happens because of the renormalization procedure, which eliminates a contribution of high energy modes and introduces some scale in the theory.

Let us demonstrate the origin of the conformal anomaly for model (8.69). The effective action of the model is

$$W[g] = \frac{1}{2}\ln\det L[g], \tag{8.72}$$

$$L[g] = -\nabla^2 + \frac{n-2}{4(n-1)}R. \tag{8.73}$$

According to (8.70), (8.71), the operator $L[g]$ transforms as

$$L[\bar{g}] = e^{\frac{n+2}{2}\sigma} L[g] e^{-\frac{n-2}{2}\sigma}. \tag{8.74}$$

To apply the results of Sect. 5.7, consider a family of operators

$$L_\alpha[g] = e^{\frac{n+2}{2}\alpha\sigma} L[g] e^{\frac{n-2}{2}\alpha\sigma}. \tag{8.75}$$

One can easily check that

$$\frac{d}{d\alpha}\mathrm{Tr}\, e^{-tL_\alpha} = -t2\,\mathrm{Tr}\big(\sigma e^{-tL_\alpha}\big). \tag{8.76}$$

By using (5.64) and (5.71) one finds the anomalous scaling of the effective action

$$W[\bar{g}] - W[g] = \int_0^1 d\alpha\, \zeta(0, L_\alpha[g], \sigma), \tag{8.77}$$

or, in infinitesimal form,

$$\delta W[g] = \zeta(0, L[g], \sigma). \tag{8.78}$$

From this transformation law and the definition of the stress-energy tensor (1.22) one obtains a modified Noether condition (8.11)

$$T^{\mu}_{\mu} = \frac{n}{2}\frac{1}{\sqrt{g}}\left(\frac{\delta\zeta(0,L,\sigma)}{\delta\sigma}\right)_{\sigma=0}. \tag{8.79}$$

The right hand side of (8.79) is called the conformal anomaly or the Weyl anomaly.

To give an example let us consider (8.69) in two dimensions where the curvature coupling is not required. Suppose that the background Riemannian manifold $\mathscr{M}$ has topology of a disc, and impose the Dirichlet condition on $\varphi$ at the boundary $\partial\mathscr{M}$. The 2d Laplacian $L = -\nabla^2$ in this case does not have zero modes. Therefore,

$$\zeta(0,L,\sigma) = a_1(\sigma, L) = \frac{1}{24\pi}\left(\int_{\mathscr{M}} \sqrt{g}\, d^2x\, R\sigma + 2\int_{\partial\mathscr{M}} \sqrt{h}\, dx\, K\sigma\right), \tag{8.80}$$

where $K$ is the trace of the extrinsic curvature of $\partial\mathscr{M}$. From (8.79) and (8.80) one finds the anomalous trace,

$$T^{\mu}_{\mu} = \frac{R}{24\pi}. \tag{8.81}$$

To get this result we have assumed that conformal variations $\sigma$ vanish on the boundary.

## 8.6 Using Anomalies to Calculate the Effective Action

Chapter 7 and the exercises there provided us with a number of examples where effective actions can be written in a form of local functionals of background fields. In general, however, the effective action, as distinct from a classical action, is essentially non-local and its calculation is rather involved. The aim of this section is to give an idea about the non-local structure of the effective action by using two-dimensional models which allow a simple analytical treatment.

The idea of the method is the following: in two-dimensional models where the background fields can be brought to some trivial configuration by a symmetry transformation the effective action is reduced to the quantum anomaly of the corresponding transformation. By the words 'some trivial configuration' one usually means either constant background fields or configurations where the effective action can be computed exactly.

**Gauge Theories** We begin with a computation of the effective action for fermions in an external Abelian gauge field $V_\mu$,

$$W[V] = -\ln\det \not{D}(V), \tag{8.82}$$

$$\not{D}(V) \equiv i\gamma^{\mu}(\partial_\mu + iV_\mu). \tag{8.83}$$

The vector potential $V_\mu$ is assumed to be real, the base background manifold is flat. As was already discussed in Exercise 3.3 any vector field in two dimensions can be represented as

$$V_\mu = \partial_\mu \varrho(x) + \varepsilon_{\mu\nu}\partial^\nu \varphi(x) + V_\mu^H. \tag{8.84}$$

The harmonic vector field $V_\mu^H$ and the scalar fields obey the equations

$$\partial^\mu V_\mu^H = \varepsilon^{\mu\nu}\partial_\nu V_\mu^H = 0, \tag{8.85}$$

$$\varrho = -\Delta^{-1}\partial_\nu V^\nu, \qquad \varphi = \frac{1}{2}\Delta^{-1}\varepsilon^{\mu\nu}F_{\mu\nu}, \tag{8.86}$$

where it is implied that $\Delta\varrho \neq 0$, $\Delta\varphi \neq 0$. It follows from (8.85) that $\Delta V_\mu^H = 0$. The field strength is $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\mu V_\nu$.

The parameter $\varrho$ determines a gauge transformation of $V_\mu$ which does not change the determinant of the Dirac operator, see Sect. 8.2. Since the effective action does not depend on $\varrho$ we put $\varrho = 0$. One can now use the property of two-dimensional gamma-matrices, $\varepsilon_{\mu\nu}\gamma^\nu = i\gamma_*\gamma^\mu$, to represent the Dirac operator as

$$\not{D}(V) = e^{-\gamma_*\varphi}\not{D}(V^H)e^{-\gamma_*\varphi}. \tag{8.87}$$

This equation can be interpreted as an Abelian chiral (axial) transformation of $\not{D}(V^H)$. By taking into account that chiral transformations are anomalous one can write

$$W[V] = W[V^H] + \mathscr{A}(\varphi), \tag{8.88}$$

where $\mathscr{A}(\varphi)$ is the two-dimensional axial anomaly. The harmonic vector field $V_\mu^H$ is an example of a trivial configuration.

The following calculation of the anomaly is based on the results of Sect. 5.7. One defines a family of Laplace-type operators

$$L_\alpha \equiv \left(e^{-\alpha\gamma_*\varphi}\not{D}(V^H)e^{-\alpha\gamma_*\varphi}\right)^2, \tag{8.89}$$

where $\alpha$ is a real parameter, and notes that

$$\frac{d}{d\alpha}\mathrm{Tr}\, e^{-tL_\alpha} = -4t\,\mathrm{Tr}\left(\varphi\gamma_\star e^{-tL_\alpha}\right). \tag{8.90}$$

By using (5.64) and (5.71) one finds

$$\mathscr{A}(\varphi) = -\frac{1}{2}(\ln\det L_1 - \ln\det L_0) = 2\int_0^1 d\alpha\,\zeta(0, L_\alpha, \gamma_\star\varphi). \tag{8.91}$$

If the contribution of zero modes in the zeta-function is neglected one can put $\zeta(0, L_\alpha, \gamma_\star\varphi) = a_2(\gamma_\star\varphi, L_\alpha) \sim \alpha$. It is straightforward to see that

$$W[V] = W[V^H] - \frac{1}{16\pi}\int d^2x\, d^2y\, F(x)\Delta^{-1}(x-y)F(y). \tag{8.92}$$

Here we have used (8.86) and put $F = \varepsilon^{\mu\nu}F_{\mu\nu}$. It is the last term in the r.h.s. (8.92) which carries non-localities typical for the effective action.

**Gravity Theories** We now consider computation of the effective action of a massless scalar field on a two-dimensional manifold $\mathscr{M}$. This is a theory with the conformal anomaly discussed in Sect. 8.5. Suppose that $\mathscr{M}$ is related by a conformal transformation to some simple manifold $\bar{\mathscr{M}}$. The metric $\bar{g}_{\mu\nu}$ on $\bar{\mathscr{M}}$ is considered as a trivial background. For example, for a manifold $\mathscr{M}$ with a boundary $\bar{\mathscr{M}}$ may be a disk. The effective action can be written as

$$W[g] = W[\bar{g}] + \mathscr{A}[\sigma, \bar{g}], \tag{8.93}$$

where $\mathscr{A}[\sigma, \bar{g}]$ is a conformal anomaly and $\sigma$ is a parameter of a conformal transformation from $\mathscr{M}$ to $\bar{\mathscr{M}}$, see (8.70). The derivation is based on formula (8.77). Equation (8.75) yields a family of operators $L_\alpha[g] = e^{2\alpha\sigma} L[g]$ which satisfy (8.76). By using (8.80), one gets

$$W[g] = W[\bar{g}] - \frac{1}{24\pi}\int_0^1 d\alpha\left(\int_{\mathscr{M}} d^2x\, \sqrt{g_\alpha} R_\alpha \sigma + 2\int_{\partial\mathscr{M}} dx\, \sqrt{h_\alpha} K_\alpha \sigma\right), \tag{8.94}$$

where $(g_\alpha)_{\mu\nu} = e^{-2\alpha\sigma} g_{\mu\nu}$, and $R_\alpha$, $K_\alpha$ are the corresponding curvatures computed for $(g_\alpha)_{\mu\nu}$,

$$R_\alpha = e^{2\alpha\sigma}\left(R + 2\alpha\nabla^2\sigma\right), \tag{8.95}$$

$$K_\alpha = e^{\alpha\sigma}(K + \alpha\sigma_{;n}), \tag{8.96}$$

see Exercise 8.5. These relations allow one to integrate in (8.94) over $\alpha$ and obtain

$$W[g] = W[\bar{g}] - \frac{1}{24\pi}\left(\int_{\mathscr{M}} d^2x\, \sqrt{g}(R\sigma - (\nabla\sigma)^2) + \int_{\partial\mathscr{M}} dx\, \sqrt{h}(2K + 3\sigma_{;n})\sigma\right). \tag{8.97}$$

A possible contribution of zero modes of the scalar Laplacian on $\mathscr{M}$ is not taken into account in (8.97). Their effect is studied in Exercise 8.6.

If $\mathscr{M}$ is conformally flat and does not have boundaries the effective action can be written in the following non-local form:

$$W[g] = \frac{1}{48\pi}\int d^2x\, \sqrt{g} d^2x'\, \sqrt{g'} R(x)\Delta^{-1}(x, x')R(x'), \tag{8.98}$$

where we used the results of Exercise 8.5, see Eq. (8.118). In a number of physical applications functional (8.98) is known as the Polyakov nonlocal action.

The effective action (8.98) can be also represented in an equivalent local form. Consider the following two-dimensional gravity theory where dynamical variables are the metric $g_{\mu\nu}$ of $\mathscr{M}$ and a scalar field $\varphi$ which is non-minimally coupled to the curvature of $\mathscr{M}$,

$$I_L[g, \varphi] = -\frac{1}{8\pi}\int_{\mathscr{M}} d^2x\, \sqrt{g}\left(-(\nabla\varphi)^2 + \frac{2}{\gamma}R\varphi + \frac{\mu}{\gamma^2}\right). \tag{8.99}$$

Such a theory is called the Liouville gravity and possesses a nontrivial dynamics. If a 'cosmological constant' vanishes, $\mu = 0$, and $\mathscr{M}$ has no boundaries, equations for the metric in the Liouville gravity (8.99) coincide with the equations which follow from (8.98) provided $\varphi$ satisfies its equations of motion as well, see Exercise 8.7. Formally the Liouville theory follows from (8.97) under the identification

$$\varphi = \frac{2}{\gamma}\sigma, \quad \gamma = \sqrt{12}. \tag{8.100}$$

The Liouville gravity is a classical conformal theory where the algebra of conformal transformations, the so-called Virasoro algebra, has a central extension with a central charge $c = 12/\gamma^2$.

## 8.7 Parity Anomaly and the Chern-Simons Action

In this Chapter we have studied quantum anomalies related to continuous symmetries. Quantum corrections may also break discrete symmetries. In this last section we briefly discuss this effect by using example of the so-called parity anomaly which may appear in any odd number of dimensions.

Let us take the Dirac operator $\not{D}$ in a gauge background $V_\mu$, see Eq. (8.83). The parity transformation is a reflection of all coordinates and components of the vector field $V_\mu$. Under this transformation all eigenvalues $\lambda_k$ of the Dirac operator change the sign. Classically, the reflection operation can be compensated by inverting the sign of the $\gamma$-matrices, i.e., by going to another representation of the Clifford algebra. In a quantum theory, if the spectrum is not symmetric, the parity symmetry may be violated in regularized quantities. In the zeta-function regularization, this happens due to the choice of the phase in front of the second sum in (5.56).

Consider the effective action of fermions $W[V]$ which is determined by (8.82). We assume that the base manifold is three-dimensional Euclidean space. The gauge field $V_\mu$ may be non-Abelian, in general. We recall formula (5.58),

$$\ln(\det \not{D})_s = -\mu^s \Gamma(s)\zeta(s, \not{D}), \tag{8.101}$$

for the zeta-regularized determinant. The zeta function itself is given by (5.56). Contrary to the situation in even dimensions considered in Sect. 5.6, in odd dimensions the spectrum of the Dirac operator may not be symmetric, and there may exist a parity-odd part of the zeta function

$$\begin{aligned} \frac{1}{2}(\zeta(s, \not{D}) - \zeta(s, \not{D}_P)) &= \frac{1}{2}(1 - e^{-i\pi s})\Bigg[\sum_{\lambda_k>0}(\lambda_k)^{-s} - \sum_{\lambda_k<0}(-\lambda_k)^{-s}\Bigg] \\ &= \frac{1}{2}(1 - e^{-i\pi s})\eta(s, \not{D}), \end{aligned} \tag{8.102}$$

where $\not{D}_P$ is the parity transformed Dirac operator, and $\eta(s, \not{D})$ is the eta-function defined in (5.34). The zero of the factor $1 - e^{-i\pi s}$ cancels the pole of the $\Gamma$-function in (8.101) at $s = 0$. Thus, the parity-odd part of the effective action is finite,

$$W^{\text{P-odd}}[B] = \frac{i\pi}{2}\eta(0, \not{D}(B)). \tag{8.103}$$

It is convenient to use an integral representation for the eta-function,

$$\eta(s, \not{D}) = \frac{2}{\Gamma((s+1)/2)} \int_0^\infty dt\, t^s \,\mathrm{Tr}\big(\not{D} e^{-t^2 \not{D}^2}\big), \tag{8.104}$$

which follows from (5.36). The variation of $\eta(s)$ with respect to $V_\mu$ in $\not{D}$ yields

$$\delta\eta(s, \not{D}) = \frac{2}{\Gamma((s+1)/2)} \int_0^\infty dt\, t^s \frac{d}{dt} \,\mathrm{Tr}\big((\delta\not{D}) t e^{-t^2 \not{D}^2}\big). \tag{8.105}$$

Now, by taking $s \to 0$ and assuming that the heat kernel decays fast enough at $t^2 \to \infty$, which is usually true, one arrives at the result

$$\begin{aligned}\delta\eta(0, \not{D}) &= -\frac{2}{\sqrt{\pi}} \lim_{t\to 0} \mathrm{Tr}\big((\delta\not{D}) t e^{-t^2 \not{D}^2}\big) \\ &= -\frac{2}{\sqrt{\pi}} \lim_{t\to 0} \mathrm{Tr}\big((\delta\not{D}) t^{1/2} e^{-t \not{D}^2}\big).\end{aligned} \tag{8.106}$$

To evaluate this limit we use heat kernel expansion (4.9). The coefficient $a_0$ does not contribute because of the $\gamma$-trace and we are left with the expression

$$\delta\eta(0, \not{D}) = -\frac{2}{\sqrt{\pi}} a_2(\delta\not{D}, \not{D}^2). \tag{8.107}$$

Next, one can use (4.126) with $Q = \delta\not{D} = -\gamma^\mu V_\mu$ and $E$ given in (12.105) with $A = 0$, calculate the gamma-matrix trace with the help of the relation

$$\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = 2i\varepsilon^{\mu\nu\rho}, \tag{8.108}$$

and obtain

$$\delta\eta(0, \not{D}) = -\frac{1}{4\pi^2} \int d^3x\, \varepsilon^{\mu\nu\rho}\, \mathrm{tr}(\delta V_\mu \cdot F_{\nu\rho}), \tag{8.109}$$

where the trace is now taken over the remaining gauge indices. This variational equation can be solved. The result up to a constant is

$$\eta(0, \not{D}) = -\frac{1}{4\pi^2} \int d^3x\, \varepsilon^{\mu\nu\rho}\, \mathrm{tr}\bigg( V_\mu \partial_\nu V_\rho + \frac{2i}{3} V_\mu V_\nu V_\rho \bigg). \tag{8.110}$$

Finally, by substituting (8.110) in (8.103), we obtain

$$W^{\text{P-odd}}[V] = -\frac{1}{2} I_{\text{CS}}[V], \tag{8.111}$$

$$I_{\text{CS}}[V] = \frac{i}{4\pi} \int d^3x\, \varepsilon^{\mu\nu\rho}\, \mathrm{tr}\bigg( V_\mu \partial_\nu V_\rho + \frac{2i}{3} V_\mu V_\nu V_\rho \bigg). \tag{8.112}$$

Functional (8.112) is called the Chern-Simons action.

The Chern-Simons action has numerous physical applications. One of them is the quantum Hall effect. To understand this phenomenon, let us take an Abelian electromagnetic gauge field $V_\mu = A_\mu$. The cubic term in (8.112) then vanishes. The

Wick rotation to physical Minkowski space results in an imaginary factor in front of the action. The electric current induced by quantum effects is

$$\langle j^\mu \rangle \sim \frac{\delta S_{\rm CS}}{\delta A_\mu} \sim \varepsilon^{\mu\nu\rho} F_{\nu\rho}.$$

Consequently, a constant electric field $E_i \propto F_{0i}$ in $2+1$ dimensions produces a constant current $j^k \propto \varepsilon^{k0i} E_i$ in the direction orthogonal to the electric field. This is what the Hall effect is about. The coefficient in front of the Chern-Simons action defines the Hall conductivity.

## 8.8 Literature Remarks

In classical theories symmetries are important for understanding different phenomena such as conservation laws, integrability of equations of motion, and others. Introduction to Noether theorems together with a historical account and further references on this subject can be found in [54]. The derivation of the first Noether theorem is given, e.g., in [40].

The chiral anomaly was discovered by Adler, Bell and Jackiw [2, 31] in 1969. This became one of very fruitful insights in properties of quantum field theories. The absence of the anomalies is an important consistency requirement for a quantum model, otherwise appearance of the anomalies must result in observable physical effects. For example, the chiral anomalies cancel out in the Standard Model of electroweak interactions while the Adler-Bell-Jackiw anomaly is vital for understanding the low-energy hadron physics.

Our focus here was on application of spectral methods for derivation of the anomalies. That is why a large portion of the known material on anomalies has not been included in this Chapter. For example, we have not discussed algebraical and topological techniques of determining anomalies, in particular how anomalies are related via so-called decent equations to characteristic classes. As well, we have not described an important observation by Fujikawa [116] that the reason for the chiral anomaly is in non-invariance of the integration measure in the path integral over fermion fields. A comprehensive discussion of these questions together with other mathematical aspects of quantum anomalies can be found in the book by Bertlmann [35]. Some other useful expositions of anomalies are [161, 192]. Spectral methods are used in finite-mode regularization approach by Andrianov and Bonora [11] which has applications to the hadron physics [12].

The gravitational anomalies in two dimensions play an important role in the string theory on the world sheet. They were first found by Alvarez-Gaume and Witten [8] by using Feynman diagrams in the linearized gravity theory. One can show that the gravitational anomalies exist in the space-times with the dimension $n = 4p + 2$ where $p = 0, 1, 2, \dots$, see [8]. The equivalence of the Lorentz and Einstein anomalies was demonstrated by Bardeen and Zumino [24]. The fact that consistent and covariant anomalies can be related by adding a local polynomial was also established in [24]. The heat kernel method was used for calculation of the gravitational anomalies in [179, 180]. An overview of various approaches to the Wick rotation of chiral fermions can be found in [240].

Conformal anomalies are more of a theoretical interest. Among their physical applications one should mention two-dimensional models where the conformal and the gravitational anomalies are related to the flux of the Hawking radiation from a black hole [48, 68, 219]. A review on 2D gravities is [149]. A historical account of works on conformal anomalies is [95].

In conformal field theories in two dimensions the conformal anomalies are related to the central extension of the Virasoro algebra of conformal transformations. The Liouville theory mentioned in Sect. 8.6 is known from 19th century as a theory of negatively curved surfaces. A review of its properties and relation to the Virasoro algebra can be found in [81, 229].

The fact that the femions in 3 dimensions generate the Chern-Simons action through the parity anomaly was discovered in mid 1980's [9, 199, 200, 216, 217], see [98] for a review. In our presentations of this topic we mostly follow [9, 76, 247].

The examples of non-localities in the effective action presented in this Chapter is just a small part of a very broad subject of techniques and approximations used to calculate the effective action. This subject goes well beyond the scope of this book. Among most important references we mention covariant perturbation theory of Barvinsky and Vilkovisky [27, 28]. Some of its applications to finite-temperature theories can be found in [151, 152].

Recommended Exercises are 8.1, 8.3, 8.6, 8.9.

## 8.9 Exercises

**Exercise 8.1** Calculate the axial anomaly for the effective action discussed in Sect. 8.2

$$W[V,A] = -\ln\det \not{D}(V,A), \tag{8.113}$$

where $\not{D}(V,A) = i\gamma^\mu(\partial_\mu + V_\mu + i\gamma_\star A_\mu)$. Consider the case of a gauge theory in two dimensions.

**Exercise 8.2** Consider a chiral theory on a flat even-dimensional manifold with an $SU(N)$ background gauge field $B_\mu$. The variation of the effective action is defined as

$$\delta W[B] = -\delta \ln\det \hat{D}(B), \tag{8.114}$$

where $\hat{D}(B)$ has a block form analogous to (8.31) where $D = D_+(B)$ is a chiral part of the operator $\not{D}(B) = i\gamma^\mu(\partial_\mu + B_\mu)$, and $\bar{D}$ does not depend on $B_\mu$. The classical theory is invariant with respect to the gauge transformations

$$\delta_\lambda B_\mu(x) = i(\partial_\mu \lambda(x) + [B_\mu(x), \lambda]), \tag{8.115}$$

where the gauge parameter $\lambda^+ = \lambda$ belongs to the Lie algebra of $SU(N)$. Find the anomalous Noether condition (8.7) for the current

$$\langle J^\mu \rangle = \frac{\delta W}{\delta B_\mu}. \tag{8.116}$$

**Exercise 8.3** In a diffeomorphism invariant gravity theory prove that the definitions of the stress-energy tensor in terms of metric and vielbein variations are equivalent,

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta I}{\delta g_{\mu\nu}}, \qquad T^a{}_\mu \equiv -e^{-1} \frac{\delta I}{\delta e_a^\mu}, \tag{8.117}$$

where $e = \det e_\mu^a = \sqrt{g}$, $g = |\det g_{\mu\nu}|$, see Eqs. (1.22) and (8.13).

**Exercise 8.4** For the operators $D_\pm$ defined in (8.42) check that $(D_+)^\dagger = D_-$.

**Exercise 8.5** Prove that (8.69) does not change under transformations of the metric and scalar fields, (8.70), (8.71). To this end, prove the transformation law

$$\bar{R} = e^{2\sigma}\big(R + (n-1)\big(2\nabla^2\sigma + (2-n)(\nabla\sigma)^2\big)\big), \tag{8.118}$$

$$\bar{K} = e^{\sigma}\big(K + (n-1)\sigma_{;n}\big), \tag{8.119}$$

where $\bar{R}$ and $R$ are curvature scalars for $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$, respectively. Infinitesimal conformal changes of some geometrical quantities are listed in (4.51), (4.52) and (4.90).

**Exercise 8.6** Calculate the effective action of a two-dimensional massless scalar field on a compact space without boundaries by taking into account the presence of zero modes of the operator $-\nabla^2$.

**Exercise 8.7** Prove that on a manifold without boundaries functionals (8.97) and (8.99) give equivalent equations for the metric.

**Exercise 8.8** The massless Dirac field is an example of a classical conformal field theory. Prove that the Dirac operator under conformal transformations of metric (8.70) changes as

$$\not{D}[\bar{g}] = e^{\frac{n+1}{2}\sigma} \not{D}[g] e^{-\frac{n-1}{2}\sigma}, \tag{8.120}$$

$$\not{D}[g] = i\gamma^\mu \nabla_\mu^{(s)} = i\gamma^\mu(\partial_\mu + w_\mu^{[s]}). \tag{8.121}$$

**Exercise 8.9** By using (8.70) calculate the trace anomaly and the effective action for a spinor field on a two-dimensional background,

$$W[g] = -\ln\det \not{D}[g], \tag{8.122}$$

where $\not{D}[g]$ is defined in (8.121).

**Exercise 8.10** Prove that Chern-Simons action (8.112) is gauge invariant up to surface terms.

# Chapter 9
# Vacuum Energy

## 9.1 The Definition

In this Chapter we take a closer look at the vacuum energy. As earlier, we restrict ourselves by free quantum fields. “Free” means here that quantum fluctuation do not have self-interactions, though the interactions with classical background fields may be rather nontrivial. Computations of the vacuum energy and first evidences for its physical importance have been exposed in Chap. 7 where we discussed properties of the Coleman-Weinberg potential in the Minkowski space-time.

The aim of the present Chapter is to introduce a number of typical physical problems and some spectral methods which allow one to do computations when background is non-trivial, as for example, when background fields are not constant, or when background manifolds have boundaries. Our only requirement is that external classical system should be static, so that we have a well defined notion of energy of quantum fluctuations and can apply results of Sect. 2.5.

Let $\omega_i$ be single-particle energies. Consider the formal expression for vacuum energy (2.49) of a Bose field. In this Chapter we use the following zeta-function regularization of equations like (2.49):

$$E_s = \frac{\mu^{2s}}{2}\sum_i (\omega_i^2)^{\frac{1}{2}-s}. \tag{9.1}$$

The physical value of vacuum energy corresponds to the limit $s \to 0$. A real parameter $\mu$ has the dimension of a mass and is introduced to keep the physical dimension of the regularized vacuum energy. In expressions like (9.1) all frequencies squared are assumed to be positive. Complex and zero $\omega_i$ correspond to modes which are not quantized, see remarks in the end of Sect. 6.2. Such energies are not included in (9.1). In this Chapter we put the Planck constant $\hbar = 1$.

In general, the single-particle energies $\omega_i$ are determined by a non-linear spectral problem like (7.41). Without loss of generality we suppose that in (7.41) the operator

D. Fursaev, D. Vassilevich, *Operators, Geometry and Quanta*, Theoretical and Mathematical Physics,
DOI 10.1007/978-94-007-0205-9_9, 

$L(\omega)$ does not depend on the spectral parameter, i.e. $\omega_i^2$ are just eigenvalues of an elliptic second order operator $L$. One may write (9.1) as

$$E_s = \frac{\mu^{2s}}{2}\,\mathrm{Tr}(L^{\frac{1}{2}-s}) = \frac{\mu^{2s}}{2}\zeta\left(-\frac{1}{2}+s, L\right). \tag{9.2}$$

If the number of space-time dimensions is $d+1$ a base manifold for $L$ is $d$-dimensional. By using Eq. (5.28) one can relate the divergent (pole) part of the vacuum energy to a heat kernel coefficient,

$$E_s = -\frac{1}{4\sqrt{\pi}}a_n(L)\frac{1}{s} + \mathscr{O}(s^0), \tag{9.3}$$

where $n = d+1$. A similar divergent term appears in the corresponding finite-temperature effective action $\ln\det P_E$, as a logarithmic divergence in (7.67), see Chap. 7. There is a relation between the heat coefficients, $a_n(L) = \sqrt{4\pi}\,a_n(P_E)/\beta$, where $P_E$ is the corresponding "wave" operator in $n$ dimensions, (7.43), and $1/\beta$ is the temperature. A direct check of this relation in a general setting is contained in Exercise 7.9. The divergences of $E_s$ and of the effective action are equivalent because the vacuum energy is a part of the effective action, see (7.59). Thus, the standard renormalization procedure eliminates the $1/s$ terms and leaves $E_s$ finite in the limit $s \to 0$.

Although (9.3) lists only logarithmic divergences (poles $1/s$) other types of divergent terms may be present in other regularizations. It may happen that some of these divergences are not removed by the standard renormalization. This can be caused by a mathematical idealization of physical conditions, which is a common problem in case of systems with boundaries, see discussion below.

## 9.2 The Casimir Effect

One of manifestations of the vacuum energy is an interaction of neutral bodies or surfaces in empty space due to quantum fluctuations. This phenomenon has an experimental confirmation and is called the Casimir effect.

Consider, for example, two parallel metal plates. Their interaction appears because the amount of energy stored in vacuum fluctuations of the electromagnetic field between the plates depends on a distance between the plates. Mathematical reasons behind that are the following. Conducting plates screen the field. The components of the electric field are decomposed on a tangential part, which is parallel to plates, and a normal part, orthogonal to the plates. If the plates are ideally conducting the tangential part of the electric field vanishes on the plates, i.e. it obeys there a Dirichlet condition. The boundary conditions affect the single-particle spectrum $\omega_i$ and, hence, vacuum energy (9.1).

We carry out the computation for a simpler situation of a real scalar field $\varphi$. The parallel plates are embedded in a three-dimensional space, the separating distance is $l$. The scalar field is assumed to satisfy the Dirichlet boundary condition on the both plates. The single-particle energies of the modes, which are solutions to the wave equation $\Box\varphi = 0$, read

$$\omega_{\mathbf{k},n} = \sqrt{\mathbf{k}^2 + \frac{\pi^2 n^2}{l^2}}, \tag{9.4}$$

where $\mathbf{k} \in \mathbb{R}^2$ is the momentum tangential to the plates, and $n = 1, 2, \dots$ is the wave number in the direction orthogonal to the plates. Since there is a translation invariance in the directions along the plates it makes sense to calculate the energy *density* per unit area of the plates, $\mathcal{E}_s = E_s/V$, where $V$ is the area of a plate and $E_s$ is defined by (9.2). In the limit $V \to \infty$ the energy density turns out to be finite and is given by the following formula:

$$\mathcal{E}_s = \frac{\mu^{2s}}{2} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{d\lambda}{4\pi} \left[ \lambda + \frac{\pi^2 n^2}{l^2} \right]^{\frac{1}{2}-s}. \tag{9.5}$$

To obtain this relation one should correctly fix the integration measure over the tangential momenta $\mathbf{k}$. This can be done in different ways. For instance, we may note that for finite $V$ the sum over discrete spectrum can be replaced by an integral with measure $dN(\lambda)$, where $N(\lambda)$ is the counting function studied in Sect. 5.4. In the considered case $\lambda = \mathbf{k}^2$ are eigenvalues a two-dimensional Laplacian. To perform the $V \to \infty$ limit, one can use (5.39) and conclude that $dN(\lambda)/d\lambda = V/(4\pi)$.

The integral over $\lambda$ in (9.5) and then the sum over $n$ are easily performed,

$$\mathcal{E}_s = \frac{\mu^{2s}}{8\pi} \frac{1}{s - 3/2} \sum_{n=1}^{\infty} \left( \frac{\pi n}{l} \right)^{3-2s} = \frac{\mu^{2s}}{8\pi} \frac{\zeta_R(2s-3)}{s - 3/2} \left( \frac{\pi}{l} \right)^{3-2s}, \tag{9.6}$$

where $\zeta_R(2s-3)$ is the Riemann zeta-function (5.5). This expression is finite at $s \to 0$, and with the help of (5.20), (5.15) we obtain

$$\mathcal{E} = \lim_{s \to 0} \mathcal{E}_s = -\frac{\pi^2}{1440 l^3}. \tag{9.7}$$

This energy density is negative. Consequently, the corresponding force between the plates, $F = \partial\mathcal{E}/\partial l$, is attractive. It is $F$ that is called the Casimir force and is measured in experiments. The Casimir force for electromagnetic field is two times the force in the scalar case.

The explanation of the fact that no poles are encountered in (9.6) when regularization is removed is the following. The divergent part of the vacuum energy in the given regularization is determined by (9.3). We need to know the heat coefficient $a_4$. The bulk part of $a_4$ vanishes because the field is massless and the base manifold is flat, see (4.58). Since the boundary is flat the only boundary invariant for this system is the area integral of the plates. This invariant enters $a_1$, see (4.71), not $a_4$. Thus, the boundary part of $a_4$ vanishes as well.

In other regularizations there may appear divergences related to $a_1$. These divergences are a mathematical artifact: in a physical world pure Dirichlet conditions cannot be realized because there cannot exist ideally conducting metal plates. Fortunately, $a_1$ does not depend on the distance between the plates. Thus, $a_1$ does not

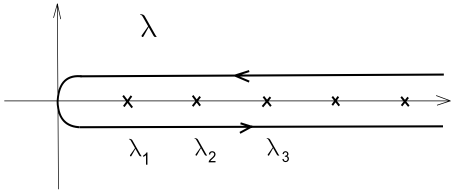


**Fig. 9.1** The integration contour $C$ in Eq. (9.9)

enter the Casimir force which is the derivative of the energy with respect to the distance, and the force is finite.

Further examples of calculation of the Casimir energy can be found in Exercises 9.1–9.3.

## 9.3 Calculations on the Complex Plane

In general, the single-particle spectrum is not known explicitly but is rather given by roots of some function. We describe a method how to do computations in this case.

The method is based on a convenient representation of the vacuum energy in terms of a contour integral in the complex plane and can be described as follows. Suppose that the single-particle spectrum is determined by the eigenvalues $\{\lambda\}$ of an operator $L$, i.e. $\omega_i^2 = \lambda$. Let the spectrum of $L$ be positive and discrete, and assume that the eigenvalues $\lambda_n$ are defined as roots of the equation

$$f(\lambda) = 0. \tag{9.8}$$

The function $\partial_\lambda \ln(f(\lambda))$ has poles with unit residues at $\lambda = \lambda_n$. Consequently, modulo some natural assumptions on the analytic properties of $f$, the zeta-function of $L$ can be represented by a contour integral

$$\zeta(s, L) = \frac{1}{2\pi i} \int_C d\lambda\, \lambda^{-s} \partial_\lambda \ln(f(\lambda)), \tag{9.9}$$

where the contour $C$ runs anticlockwise around the positive semiaxis, see Fig. 9.1. Once the integral definition of $\zeta(s, L)$ is known one can use (9.2) for computation of $E_s$.

To illustrate the method, let us consider computation of the vacuum energy for a smooth static potential $V(x)$ in $1+1$ dimensions. We take a one-dimensional operator

$$L = -\partial_x^2 + m^2 + V(x) \tag{9.10}$$

and assume that $V(x)$ vanishes fast enough as $x \to \pm\infty$. Any constant part of the potential may be absorbed in $m^2$. There are two types of eigenmodes of the operator $L$. The discrete spectrum is formed by the bound states with the eigenfrequencies $\omega_i^2 = m^2 - \kappa_i^2$. The corresponding eigenmodes behave as $e^{\pm\kappa_i x}$ at $x \to \mp\infty$, thus decay at infinity and have a finite $\mathrm{L}^2$ norm. For quantum stability of the system (to avoid imaginary single-particle energies) the mass should be sufficiently large, $m \geq \kappa_i$ for all bound states. For a smooth potential, the number of bound states is finite. There is also continuum spectrum with the single-particle energies $\omega^2(k) = m^2 + k^2$. Corresponding modes oscillate at large distances as $e^{\pm ikx}$. They have an infinite $L^2$ norm and are normalized to a delta-function in the momentum space.

The vacuum energy has two separate contributions $E_B$ and $E_C$, from the bound states and the continuum spectrum, respectively,

$$E = E_B + E_C. \tag{9.11}$$

The bound state part is given by a finite sum

$$E_B = \frac{1}{2}\sum_i (m^2 - \kappa_i^2)^{1/2} \tag{9.12}$$

and is convergent, so that no regularization is needed for this part.

The continuum spectrum part is given by a momentum integral which is usually divergent and has to be regularized. To define a zeta-regularized expression and to find a form of the spectral density, it is convenient to introduce boundaries at $x = \pm l$ with some large $l$ which will be sent to infinity at the end of the calculations.

Without boundaries, for each momentum $k$ there are two independent solutions $\eta_1$, $\eta_2$ of the wave equation with the asymptotic behavior

$$\begin{aligned} &\eta_1 \sim e^{ikx} + s_{12}e^{-ikx}, \quad \eta_2 \sim s_{22}e^{-ikx} \quad \text{for } x \to -\infty, \\ &\eta_1 \sim s_{11}e^{ikx}, \quad \eta_2 \sim s_{21}e^{ikx} + e^{-ikx} \quad \text{for } x \to \infty. \end{aligned} \tag{9.13}$$

The entries of the scattering matrix $s_{ij}$ depend on the momentum $k$. Let us assume for simplicity that the potential $V$ is symmetric. Then $s_{11} = s_{22}$ and $s_{21} = s_{12}$. One can compose symmetric and antisymmetric solutions whose asymptotics at $x \to \pm\infty$ are $(s_{11} + s_{21})e^{ik|x|} + e^{-ik|x|}$ and $\pm((s_{11} - s_{21})e^{ik|x|} - e^{-ik|x|})$, respectively. Now, let us impose Dirichlet boundary conditions $\eta(x = \pm l) = 0$ for some very large $l$, so that we are allowed to substitute the asymptotic behavior of the solutions in those conditions. The condition that either symmetric or antisymmetric solution vanishes at the boundary can be expressed through a single equation

$$f(k) = ((s_{11} + s_{21})e^{ikl} + e^{-ikl})((s_{11} - s_{21})e^{ikl} - e^{-ikl}) = 0, \tag{9.14}$$

which selects the spectrum of wave numbers $k$. This spectrum is discrete. We already know how one defines the regularized vacuum energy in the discrete spectrum, $E_C(l, s) = \frac{1}{2}\sum(k^2 + m^2)^{\frac{1}{2}-s}$, where the sum is extended to all positive solutions of (9.14). As explained above, we use an integral representation

$$E_C(l, s) = \frac{1}{2}\oint \frac{dk}{2\pi i}(k^2 + m^2)^{\frac{1}{2}-s}\frac{\partial}{\partial k}\ln f(k). \tag{9.15}$$

The integration contour consists of one branch at $k = \Re k + i\epsilon$, a second branch at $k = \Re k - i\epsilon$, and a small segment $-\epsilon \leq \Im k \leq \epsilon$ along the imaginary axis. Along the upper part of the contour we keep in $f(k)$ only the terms with $\exp(-ikl)$ since

$\exp(ikl)$ vanishes as $l \to \infty$. Along the lower part of the contour we retain $\exp(ikl)$. The contribution from the third part can be dropped. One has then

$$E_C(l,s) = \frac{1}{2}\int_0^\infty \frac{dk}{2\pi i}(k^2+m^2)^{\frac{1}{2}-s}\frac{\partial}{\partial k}(4ikl + \ln(s_{11}^2 - s_{21}^2)). \tag{9.16}$$

We have included boundaries, and they may carry vacuum energy not related to the potential $V$. Now, we must subtract this energy from (9.16). This is the vacuum energy of free scalar fields ($V = 0$) of the same mass $m$ obeying the same Dirichlet boundary conditions. For such fields the scattering matrix is trivial, $s_{12} = s_{21} = 0$, $s_{11} = s_{22} = 1$, so that the subtraction of corresponding vacuum energy is equivalent to dropping the term $4ikl$ in (9.16). Now we can take the limit $l \to \infty$ to obtain

$$E_C(s) = \frac{1}{2}\int_0^\infty \frac{dk}{2\pi i}(k^2+m^2)^{\frac{1}{2}-s}\frac{\partial}{\partial k}\ln(s_{11}^2 - s_{21}^2). \tag{9.17}$$

From the scattering theory we know that

$$s_{11}^2 - s_{21}^2 = e^{2i\varsigma(k)}, \tag{9.18}$$

where $\varsigma(k)$ is the phase shift. We have,

$$E_C(s) = \frac{1}{2\pi}\int_0^\infty dk(k^2+m^2)^{\frac{1}{2}-s}\partial_k\varsigma(k). \tag{9.19}$$

We can rewrite this formula differently,

$$E_C(s) = \frac{1}{2}2\int_0^\infty dk(k^2+m^2)^{\frac{1}{2}-s}\rho(k). \tag{9.20}$$

The multiplier of $1/2$ above is the usual prefactor in the vacuum energy, the factor of 2 appears due to the symmetry of the spectrum with respect to reversing the sign of the momentum, and

$$\rho(k) = \frac{1}{2\pi}\partial_k\varsigma(k) \tag{9.21}$$

is the *spectral density* introduced in Sect. 5.4.

There is an important particular case of reflectionless potentials, $s_{12} = s_{21} = 0$. For such potentials

$$\varsigma(k) = i\sum_i \ln\frac{k - i\kappa_i}{k + i\kappa_i} = 2\sum \arctan\left(\frac{\kappa_i}{k}\right), \tag{9.22}$$

where $\kappa_i$ are the same numbers which characterize the bound state energies, see (9.12). For such potentials the condition (9.14) which defines the spectrum looks particularly simple

$$\sin(2kl + \varsigma(k)) = 0. \tag{9.23}$$

Let us suppose that we deal with a reflectionless potential and calculate the contribution from a *single* bound state $\kappa$ to the vacuum energy. The corresponding spec-

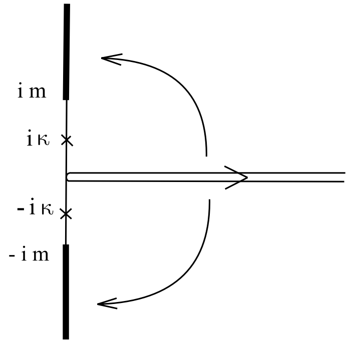


**Fig. 9.2** The integration path in (9.25) is represented by a *double horizontal line*. The integral is split into two identical parts, and the corresponding paths are then rotated in opposite directions towards the imaginary axis

tral density reads

$$\bar{\rho}(k) = -\frac{\kappa}{\pi(k^2+\kappa^2)}. \tag{9.24}$$

(Here and in what follows we put a bar over contributions of the single bound state.) This spectral density is negative. The reason is that the spectral densities we consider are not full spectral densities but rather their differences with respect to that for free fields. The corresponding continuous spectrum contribution to the vacuum energy is given by

$$\bar{E}_C(s) = -\frac{\kappa}{\pi}\int_0^\infty dk(k^2+m^2)^{\frac{1}{2}-s}\frac{1}{k^2+\kappa^2}. \tag{9.25}$$

This integral can be evaluated in its present form, but we prefer to make an analytic continuation to imaginary momenta. Continuation to imaginary momenta yields an integral which converges faster and is more suitable for numerical evaluation.

The integrand in (9.25) has two simple poles at $k = \pm i\kappa$ and two branch cuts starting at $k = \pm im$, see Fig. 9.2. We divide the integral (9.25) in two equal parts, and turn the integration path in one of them upwards, $k \to iq$, and in the other—downwards, $k \to -iq$. One should be careful with the phase factors appearing due to the continuation. For $q < m$ the multiplier $(k^2+m^2)^{\frac{1}{2}-s}$ is replaced by $(m^2-q^2)^{\frac{1}{2}-s}$ for both directions of the rotation. For $q > m$, the bracket $(k^2+m^2)^{\frac{1}{2}-s}$ receives the phase $(q^2-m^2)^{\frac{1}{2}-s}e^{i\pi(\frac{1}{2}-s)}$ when rotated upwards, and the phase $(q^2-m^2)^{\frac{1}{2}-s}e^{-i\pi(\frac{1}{2}-s)}$ when rotated downwards. Next we take the integral along the negative imaginary semi-axis and change there $q \to -q$. As a result, the contributions from $q < m$ almost cancel each other up to a contribution from the pole, which is finite at $s \to 0$, so that we can remove the regularization there

$$\bar{E}_{\text{pole}} = -\frac{1}{2}(m^2-\kappa^2)^{\frac{1}{2}}. \tag{9.26}$$

Remarkably, this term cancels precisely the corresponding contribution from the discrete spectrum, see (9.12),

$$\bar{E}_{\text{pole}} + \bar{E}_B = 0. \tag{9.27}$$

The rest of the integral yields

$$\bar{E}(s) = \bar{E}_C(s) + \bar{E}_B = \frac{\cos(\pi s)}{\pi} \int_m^\infty dq (q^2 - m^2)^{\frac{1}{2}-s} \frac{\kappa}{\kappa^2 - q^2}. \tag{9.28}$$

This integral is divergent at $s \to 0$. To isolate the divergence, let us add and subtract a term,

$$\frac{\kappa}{\kappa^2 - q^2} = \left( \frac{\kappa}{\kappa^2 - q^2} + \frac{\kappa}{q^2} \right) - \frac{\kappa}{q^2}. \tag{9.29}$$

The integral of the term in the brackets is convergent, so that we can put $s = 0$ there and evaluate the corresponding contribution

$$\bar{E}_{\text{fin}} = \frac{1}{\pi} \left[ \sqrt{m^2 - \kappa^2} \arcsin(\kappa/m) - \kappa \right]. \tag{9.30}$$

The last term on the right hand side of (9.29) produces a divergent contribution to the vacuum energy

$$\bar{E}_{\text{div}}(s) = -\frac{\kappa \cos(\pi s)}{\pi^{3/2}} m^{-2s} \Gamma\left(\frac{3}{2} - s\right) \Gamma(s). \tag{9.31}$$

To obtain a finite result we have to get rid of the pole in $\bar{E}_{\text{div}}$. The standard renormalization procedure cannot be applied in the absence of a corresponding classical theory. To resolve this problem we use additional physical arguments. It is natural to require that quantum fluctuations become effectively frozen at very large mass of the field. Thus, quantum corrections must disappear when $m$ is infinite. In practice, this means that we have to discard all terms in the vacuum energy which do not vanish in the limit $m \to \infty$.

The part $\bar{E}_{\text{fin}}$ vanishes as $m \to \infty$ and satisfies the requirement. The part $\bar{E}_{\text{div}}$ near $s = 0$ contains a pole term proportional to $m$, and finite additions proportional to $m$ and $\ln m$. Therefore, $\bar{E}_{\text{div}}(s)$ should be subtracted completely from the vacuum energy.

We accept for the time being that (9.30) represents a correct contribution of a single bound state to the vacuum energy. A comparison of this scheme to a renormalization procedure is given below as an example of quantum corrections to the kink mass. Here we stress that validity of this or other subtraction prescription must be checked separately for each physical system in question.

## 9.4 Quantization of a Kink

Let us now apply the methods of the previous section to a concrete system. We consider a $\varphi^4$ model in $1 + 1$ dimensions. The classical action reads

$$I = -\frac{1}{2} \int dt\, dx \left( (\partial_\mu \varphi)^2 + \frac{\lambda}{2} (v_0^2 - \varphi^2)^2 \right). \tag{9.32}$$

Here $\lambda$ and $v_0$ are positive constants. The equations of motion following from this action admit the famous kink solution

$$\phi_{\text{kink}} = v_0 \tanh\left( v_0 \sqrt{\frac{\lambda}{2}} (x - x_k) \right). \tag{9.33}$$

The integration constant $x_k$ describes the position of the kink. For the sake of simplicity, we put $x_k = 0$ in what follows. Solution (9.33) interpolates between two minima of the potential with $x = \pm v_0$. The mass of the kink

$$M_{\text{kink}} = \frac{v_0^3 \sqrt{2\lambda}}{3} \tag{9.34}$$

can be defined as the value of the classical Hamiltonian for theory (9.32),

$$H = \frac{1}{2} \int dx \left( (\partial_x \varphi)^2 + \pi_\varphi^2 + \frac{\lambda}{2} (v_0^2 - \varphi^2)^2 \right), \tag{9.35}$$

calculated at $\varphi = \phi_{\text{kink}}$.

One can represent $\varphi = \phi_{\text{kink}} + \chi$ and quantize small fluctuations $\chi$ by considering the kink as a background. Such a procedure is sometimes called the quantization of the kink. The vacuum energy of $\chi$ determines a quantum correction to the mass of the kink.

To study the spectrum of the fluctuations we expand action (9.32) in $\chi$. The linear in $\chi$ part vanishes due to the equations of motion, while for the quadratic part one gets

$$I_2 = -\frac{1}{2} \int dt\, dx \big( (\partial_\mu \chi)^2 + \lambda \chi^2 (-v_0^2 + 3\phi_{\text{kink}}^2) \big). \tag{9.36}$$

Variation of $I_2$ results in the following equation:

$$\big[ -\partial_0^2 + \partial_x^2 + \lambda v_0^2 \big( 1 - 3 \tanh^2 (v_0 \sqrt{\lambda/2} x) \big) \big] \chi = 0. \tag{9.37}$$

The asymptotic value of the potential in (9.37) defines the effective mass of quantum fluctuations

$$m^2 = 2\lambda v_0^2. \tag{9.38}$$

After subtracting the mass the potential in (9.37) gets in the family of so-called modified Pöschl-Teller potentials. The corresponding scattering amplitudes can be found in Problem 38 of [110], where one can learn that this is a reflectionless potential with two bound states

$$\kappa_1 = m, \quad \kappa_2 = m/2. \tag{9.39}$$

The first bound state is the translational zero mode with vanishing frequency. It corresponds to rigid translations of the kink.

One can now add up two contributions corresponding to $\kappa = \kappa_1$ and $\kappa = \kappa_2$ in (9.30) to obtain the mass shift of the kink:

$$\Delta H = m \left( \frac{1}{4\sqrt{3}} - \frac{3}{2\pi} \right) = \sqrt{2\lambda} v_0 \left( \frac{1}{4\sqrt{3}} - \frac{3}{2\pi} \right). \tag{9.40}$$

Let us emphasize that what we sum up are not contributions of the bound states to the formula like (9.12) (these are already canceled against the contributions of the poles in the spectral density), but rather contributions of bound states to the phase shift in the continuum spectrum for a reflectionless potential.

## 9.5 Supersymmetric Models

In the remaining part of this Chapter we discuss so-called supersymmetric models. The supersymmetry (SUSY) is a certain symmetry between bosons and fermions. Although SUSY is not yet discovered in Nature, it has interesting mathematical features. In particular, SUSY leads to many cancellations between bosonic and fermionic contributions to the vacuum energy and makes calculations of quantum corrections especially simple.

In this section we introduce some basic elements by using an example of $N = 1$ supersymmetric model in $1+1$ dimensions. The model is described by the action:

$$I = -\frac{1}{2}\int dt\, dx \big((\partial_\mu\varphi)^2 + U'(\varphi)\bar{\psi}\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi - 2FU - F^2\big). \quad (9.41)$$

Here $\varphi$ is a real scalar field, and $\psi$ is a Majorana spinor. We take $\gamma$-matrices in the Majorana representation

$$\gamma^0 = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^x = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (9.42)$$

In this representation the components of $\psi$ are real, hence, $\bar{\psi} = i\psi^T\gamma^0$. We use the following notation for components of a two-component spinor:

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.$$

Theory (9.41) is specified by a function $U(\varphi)$ called a superpotential. A concrete form $U(\varphi)$ is not fixed. The field $F$ is a non-dynamical degree of freedom. It can be removed from the action by means of its equation of motion,

$$F = -U(\varphi). \quad (9.43)$$

That is, $F$ is an auxiliary field.

Supersymmetry is defined as a set of transformations between scalar and spinor fields (between bosonic and fermionic degrees) which leave action (9.41) invariant. The SUSY transformations are

$$\delta\varphi = \bar{\epsilon}\psi, \qquad \delta\psi = (\gamma^\mu\partial_\mu\varphi + F)\epsilon, \qquad \delta F = \bar{\epsilon}\gamma^\mu\partial_\mu\psi, \quad (9.44)$$

where the SUSY parameter $\epsilon$ is a constant Majorana spinor.

Let us introduce some more notions. Since transformations (9.44) are parametrized by a single Majorana spinor they are $N = 1$ SUSY. Since this spinor does not depend on the space-time coordinates this SUSY is called rigid.

The invariance with respect to transformations (9.44) does not require equations of motion. One calls them 'off-shell' transformations. If $F$ is excluded from (9.41) by means of (9.43) one arrives at the action

$$\tilde{I} = -\frac{1}{2}\int dt\,dx\big((\partial_\mu\varphi)^2 + U'(\varphi)\bar{\psi}\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi + U^2\big). \tag{9.45}$$

This functional is invariant under the SUSY transformations

$$\tilde{\delta}\varphi = \bar{\epsilon}\psi, \qquad \tilde{\delta}\psi = (\gamma^\mu\partial_\mu\varphi - U)\epsilon, \tag{9.46}$$

provided that the equations of motion are satisfied. The equations of motion for (9.45) are

$$\partial^2\varphi - \frac{1}{2}U''(\varphi)\bar{\psi}\psi - UU' = 0, \tag{9.47}$$

$$(\not\partial + U'(\varphi))\psi = 0. \tag{9.48}$$

where $\not\partial \equiv \gamma^\mu\partial_\mu$. Transformations (9.46) are called 'on-shell'.

By following a general method discussed in Sect. 8.1 one can define a Noether current associated to the supersymmetry,

$$j_\mu = -(\not\partial\varphi + U(\varphi))\gamma_\mu\psi. \tag{9.49}$$

It is called the supercurrent. Since $j_\mu$ is divergence free, $\partial_\mu j^\mu = 0$, and has two spinorial components, there are two Noether charges

$$Q_\pm = \int_{-\infty}^{+\infty} dx\, j^0_\pm. \tag{9.50}$$

The charges are conserved on-shell.

Our next step is to introduce an algebra associated to the SUSY. Since the considered theory is classical we define the canonical structure and use the canonical brackets. Let us rewrite action (9.45) in a Hamiltonian form (in the notations of [158])

$$\begin{aligned}\tilde{I} &= \int dt\,dx\Big(-\frac{i}{2}(\dot{\psi}_+\psi_+ + \dot{\psi}_-\psi_-) + \frac{1}{2}(\dot{\varphi}\pi_\varphi - \dot{\pi}_\varphi\varphi) - \mathscr{H}\Big)\\ &\equiv \int dt\,dx\Big(-\frac{1}{2}(C^{-1})^{AB}\dot{z}_A\cdot z_B - \mathscr{H}\Big).\end{aligned} \tag{9.51}$$

Here $\pi_\varphi = \dot{\varphi}$ is the canonical momentum (2.30) and $\{z_A\} \equiv \{\varphi, \pi_\varphi, \psi_+, \psi_-\}$. The Hamiltonian density

$$\mathscr{H} = \frac{1}{2}\big((\partial_x\varphi)^2 + \pi_\varphi^2 + U^2 + U'\bar{\psi}\psi + \bar{\psi}\gamma^x\partial_x\psi\big) \tag{9.52}$$

does not contain time derivatives. The canonical bracket of two functionals $F_1(z)$ and $F_2(z)$ is introduced as

$$\{F_1, F_2\}_C = \int dx\,dy \sum_{AB} \frac{\delta^{(r)}F_1}{\delta z_A(x)} C_{AB} \frac{\delta^{(l)}F_2}{\delta z_B(y)}\delta(x-y), \tag{9.53}$$

where the matrix $C_{AB}$ is fixed in (9.51). The derivatives $\delta^{(r)}$ and $\delta^{(l)}$ are right and left variational derivatives. For practical use, this means that before taking the derivative one has to bring $z_A$ in $F_1$ to the rightmost position changing the sign whenever a fermionic variable is commuted through another fermionic variable. Likewise, $z_B$ in $F_2$ has to be moved to the leftmost position. In particular,

$$\{z_A(t,x), z_B(t,x')\}_C = C_{AB}\delta(x-x'),$$

and, more explicitly,

$$\{\varphi(t,x), \pi_\varphi(t,x')\}_C = \delta(x-x'), \tag{9.54}$$

$$\{\psi_\pm(t,x), \psi_\pm(t,x')\}_C = -i\delta(x-x'). \tag{9.55}$$

We use a subscript $C$ to avoid confusions with anti-commutators of operators. Note that the above canonical brackets are subject to the so-called grading rules, i.e., the bracket of two fermions is symmetric, while the bracket of a boson with either boson or fermion is antisymmetric. Upon quantization the canonical brackets are replaced by commutators or anti-commutators, and the right hand side of (9.55) is multiplied by $i\hbar$. This is consistent with the approach of Sect. 2.4, cf. Eq. (2.29).

The on-shell SUSY transformations (9.46) can be written in terms of the brackets with supercharges (9.50),

$$\tilde{\delta} z_A = -\{\bar{\epsilon}Q, z_A\}_C. \tag{9.56}$$

One says that the SUSY transformations are generated by the supercharges.

Consider now static ($\dot{\varphi}=0$) bosonic ($\psi=0$) configurations. For such configurations the Hamiltonian, which is the space integral of $\mathscr{H}$, see (9.52), is

$$H = \frac{1}{2}\int dx\big((\partial_x\varphi)^2 + U(\varphi)^2\big) = \frac{1}{2}\int dx\big((\partial_x\varphi \pm U)^2 \mp 2\partial_x\varphi U(\varphi)\big). \tag{9.57}$$

Let us introduce a function $W(\varphi)$ such that

$$W'(\varphi) = U(\varphi). \tag{9.58}$$

If we note $\partial_x\varphi U(\varphi) = \partial_x W(\varphi)$ the Hamiltonian (9.57) becomes

$$H = \frac{1}{2}\int dx\big(\partial_x\varphi \pm U\big)^2 \mp \big[W(+\infty) - W(-\infty)\big]. \tag{9.59}$$

Since the integrand in the equation above is non-negative, one concludes that

$$H \geq |W(+\infty) - W(-\infty)|, \tag{9.60}$$

and the equality is achieved if and only if

$$\partial_x\varphi \pm U(\varphi) = 0. \tag{9.61}$$

Inequality (9.60) is called the Bogomolny-Prasad-Sommerfield (BPS) bound [41, 210], and (9.61) is called the Bogomolny equation.

Let us differentiate (9.61) with respect to $x$

$$0 = \partial_x^2\varphi \pm U'(\varphi)\partial_x\varphi = \partial_x^2\varphi - U'(\varphi)U(\varphi), \tag{9.62}$$

where at the last step we used again the Bogomolny equation. One immediately recognizes the equation of motion for static bosonic configurations following from (9.45). Therefore, the Bogomolny equation implies the equation of motion.

Static bosonic configurations satisfying the Bogomolny equation are called the BPS states. An interesting feature of the BPS states is that they preserve "1/2" of the supersymmetries of the model. Indeed, since $\psi = 0$, SUSY variations of the scalar field vanish automatically, $\delta\varphi = 0$. SUSY variation of the fermionic field yields

$$\delta\psi = \begin{pmatrix} \partial_x\varphi - U & 0 \\ 0 & -\partial_x\varphi - U \end{pmatrix} \epsilon. \tag{9.63}$$

This variation vanishes if we take either the upper sign in (9.61) and $\epsilon_+ = 0$, or the lower sign in (9.61) and $\epsilon_- = 0$.

Let us now fix some classical background field $\phi(x)$. We suppose that the superpotential $U$ has several (at least two) zeros, so that $U^2$ has at least two minima. We consider a static solution $\phi(x)$ to the Bogomolny equation

$$\partial_x\phi - U(\phi) = 0, \tag{9.64}$$

where we fixed one of two possible signs for definiteness. This solution is invariant under the SUSY transformations with an arbitrary real parameter $\epsilon_+$ and $\epsilon_- = 0$. The energy of the solution is finite

$$H(\phi) = W(\phi(+\infty)) - W(\phi(-\infty)) \equiv Z_b. \tag{9.65}$$

Therefore, such a solution is a supersymmetric soliton.

One can study the canonical algebra of $\epsilon_+$ SUSY transformations. Since these transformations are generated by the supercharge $Q_-$ it is enough to calculate the canonical bracket of two $Q_-$. A straightforward computation yields

$$\{Q_-, Q_-\}_C = -2i(H - Z), \qquad Z = Z_b(\phi) + Z_f(\psi), \tag{9.66}$$

where $Z_b$ is defined above in (9.65), $H$ is the full Hamiltonian following from (9.52), and

$$Z_f = \frac{1}{4}\int dx\, \partial_x(\bar{\psi}\psi). \tag{9.67}$$

The quantity $Z$ is called the topological charge. It appears in the SUSY algebra in a topologically non-trivial situations. The charge $Z$ belongs to the center of the algebra, and is also called the central charge. The invariance of a state with respect to the $\epsilon_+$ transformation implies that this state is annihilated by $Q_-$, and, due to (9.66) also yields the BPS bound on the mass of the state.

To give a simplest example one can consider a supersymmetric extension of the $\varphi^4$ theory (9.32). In this case

$$U(\varphi) = \sqrt{\frac{\lambda}{2}}(v_0^2 - \varphi^2), \tag{9.68}$$

$$W(\varphi) = \sqrt{\frac{\lambda}{2}}\left(v_0^2\varphi - \frac{1}{3}\varphi^3\right). \tag{9.69}$$

Equation (9.64) is easily integrated yielding kink solution (9.33).

## 9.6 Quantum Corrections to Supersymmetric Solitons

The aim of this section is to quantize small fluctuations above supersymmetric solitons $\phi$ in two dimensions and derive an analytic formula for the shift of the soliton mass for any superpotential.

We start with the fermionic fluctuations. The linearized Dirac equation

$$\gamma^\mu \partial_\mu \psi + U'(\phi)\psi = 0$$

in components reads

$$\begin{pmatrix} \partial_x + U'(\phi) & -\partial_0 \\ \partial_0 & -\partial_x + U'(\phi) \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0. \tag{9.70}$$

Since the background is static, one can substitute the single-particle modes $\psi_\pm(t,x) = e^{i\omega_f t}\psi_\pm(\omega_f, x)$ in (9.70) and get

$$\begin{aligned} i\omega_f \psi_+(\omega_f, x) &= (\partial_x - U'(\phi))\psi_-(\omega_f, x), \\ i\omega_f \psi_-(\omega_f, x^1) &= (\partial_x + U'(\phi))\psi_+(\omega_f, x). \end{aligned} \tag{9.71}$$

By iterating these equations one obtains

$$\begin{aligned} \omega_f^2 \psi_+(\omega_f, x) &= -D_- D_+ \psi_+(\omega_f, x), \\ \omega_f^2 \psi_-(\omega_f, x) &= -D_+ D_- \psi_-(\omega_f, x), \end{aligned} \tag{9.72}$$

where

$$D_\pm = \partial_x \pm U'(\phi). \tag{9.73}$$

The operators $D_\pm$ are precisely the same as considered in Sect. 4.5, see (4.77). The isospectrality, the intertwining relations (4.78), and other important properties were already established there. We conclude that the eigenvalues $\omega_f^2$ in (9.72) are non-negative and identical for $\psi_\pm$, see also Sect. 5.9.

Let us now turn to the bosonic fluctuations. We represent $\varphi = \phi + \chi$, and expand Eq. (9.62) up to the linear order in the fluctuation $\chi$,

$$-\partial^2\chi + [U'(\phi)U'(\phi) + U(\phi)U''(\phi)]\chi = 0. \tag{9.74}$$

This yields the eigenvalue problem for the bosonic single-particle modes $\chi(t,x) = e^{i\omega_b t}\chi(\omega_b, x)$,

$$\omega_b^2 \chi(\omega_b, x) = -D_+ D_- \chi(\omega_b, x). \tag{9.75}$$

The Bogomolny equation (9.64) has been used here to write

$$U(\phi)U''(\phi) = U''(\phi)\partial_x\phi = \partial_x(U'(\phi)).$$

By comparing (9.75) and (9.72) one concludes that non-zero bosonic and fermionic single-particle energies, $\omega_b$ and $\omega_f$ coincide. This fact is a direct consequence of the supersymmetry.

Since bosonic and fermionic contributions to the vacuum energy come with opposite signs one would conclude that the vacuum energy of the fluctuations in the supersymmetric models is identically zero. In general, this conclusion is not correct because it does not take into account the divergences of the vacuum energy and the need to work with regularized quantities.

Consider the zeta-function regularization of vacuum energy (9.2). One can hope that some cancellations between bosonic and fermionic contributions hold for the regularized energies. However, this requires at least a discrete spectrum. Thus, one has to introduce boundaries and suitable boundary conditions. Suppose that the soliton $\phi(x)$ is localized somewhere near $x = 0$. Typically $\phi(x)$ approaches its asymptotic values exponentially fast as $x \to \pm\infty$. We put boundaries at $x = \pm l$. Since all quantum fluctuations in these models are typically massive, one may expect that for $l$ large enough quantum effects caused by the presence of boundaries decouple from the effects related to the soliton. In this case a concrete form of the boundary conditions is not relevant and one may choose any set of consistent boundary conditions which is convenient.

As we know from Sect. 3.2 admissible local boundary conditions for spinors in Minkowski space read

$$\left(1 \pm \gamma^\mu n_\mu\right)\psi|_{\partial\mathscr{M}} = 0. \tag{9.76}$$

Here $n^\mu$ is an inward pointing unit normal to the boundary. On the right boundary, $x = l/2$, this vector has the components $n^\mu = (0, -1)$, while on the left boundary it is $n^\mu = (0, 1)$. Therefore, if the signs in (9.76) are different on the left and on the right, there are two possible choices: either $\psi_+|_{\partial\mathscr{M}} = 0$ or $\psi_-|_{\partial\mathscr{M}} = 0$.

Let us study the first opportunity, $\psi_+|_{\partial\mathscr{M}} = 0$. From the first of Eqs. (9.71) we find that $\psi_-$ should satisfy Robin boundary conditions, $D_-\psi_-|_{\partial\mathscr{M}} = 0$. Since the single-particle energies for $\psi_-$ and $\chi$ are defined by the same operators $-D_+D_-$ (cf. (9.72) and (9.75)) one should impose Robin boundary conditions also on $\chi$. One has, therefore, the two sets of boundary conditions

$$\begin{aligned} \text{set A:} \qquad & \psi_+|_{\partial\mathscr{M}} = 0, \qquad (\partial_x - U'(\phi))\psi_-|_{\partial\mathscr{M}} = 0, \\ & (\partial_x - U'(\phi))\chi|_{\partial\mathscr{M}} = 0 \end{aligned} \tag{9.77}$$

and

$$\begin{aligned} \text{set B:} \qquad & \psi_-|_{\partial\mathscr{M}} = 0, \qquad (\partial_x + U'(\phi))\psi_+|_{\partial\mathscr{M}} = 0, \\ & \chi|_{\partial\mathscr{M}} = 0. \end{aligned} \tag{9.78}$$

One can also impose different boundary conditions on different components of the boundary, but we will not use this option. One can easily show that the both sets of boundary conditions are supersymmetric, see Exercise 9.5.

Up to obvious redefinitions, boundary conditions (9.77) and (9.78) coincide with the boundary conditions used in Sect. 4.5, see (4.81) and (4.82). It was demonstrated there that the operators $D_+D_-$ and $D_-D_+$ are isospectral on the interval $[-l, l]$ up to zero modes. The regularized vacuum energy of this system vanishes,

$$\Delta H_s^{\text{tot}} = \frac{\mu^{2s}}{2}\left(\sum_{\omega_b \neq 0} \omega_b^{1-2s} - \sum_{\omega_f \neq 0} \omega_f^{1-2s}\right) = 0. \tag{9.79}$$

This does not, however, mean that the mass shift of the soliton is zero. For a sufficiently large $l$ total vacuum energy consists of two parts,

$$\Delta H^{\text{tot}} = \Delta H^{\text{bou}} + \Delta H^{\text{sol}}. \tag{9.80}$$

Thus, the vacuum energy $\Delta H^{\text{sol}}$ associated with the soliton is precisely compensated by the vacuum energy $\Delta H^{\text{bou}}$ localized near the boundaries. To calculate the mass shift of the soliton one should simply calculate $\Delta H^{\text{bou}}$ and reverse the sign.

Near the boundaries and far away form the soliton one can use some effective field theory. This is a theory of free fields with the mass given by asymptotic value of the potential, and the boundary conditions defined by asymptotic values of the superpotential through (9.77) or (9.78).

Consider a free massive scalar field on an interval $[-l, l]$ subject to the Robin boundary conditions

$$(\partial_x + \mathscr{S}_1)\chi|_{x=-l} = (-\partial_x + \mathscr{S}_2)\chi|_{x=l} = 0 \tag{9.81}$$

with arbitrary $\mathscr{S}_1$ and $\mathscr{S}_2$. Generic oscillating solution can be represented as $A\sin(kx) + B\cos(kx)$. By substituting this solution into boundary conditions (9.81), one obtains the following condition which defines the spectrum:

$$\sin(2kl + \alpha_1 + \alpha_2) = 0, \tag{9.82}$$

$$\alpha_{1,2} = \arctan(\mathscr{S}_{1,2}/k). \tag{9.83}$$

Condition (9.82) coincides with (9.23) if one identifies $\varsigma(k)$ with $\alpha_1 + \alpha_2$. By comparing Eq. (9.83) with (9.22) we arrive at a remarkable result: the phase shift produced by a Robin boundary condition coincides with 1/2 of the phase shift of a reflectionless potential with a bound state $\kappa = \mathscr{S}$. Consequently, the same relation holds true for the corresponding vacuum energies.

Note, that we do not have to worry about the spectrum with imaginary $k$ since corresponding contributions are canceled against the poles of the spectral density, as we have already seen in Sect. 9.3. On can also see, that Dirichlet boundary condition produces zero phase shift in the corresponding eigenvalue equation and does not contribute to the vacuum energy. This justifies that we did not ascribe any vacuum energy to the Dirichlet boundaries in the previous section.

Let us now apply this result to the effective field theory near the boundaries of the soliton. As $x \to \pm\infty$ the classical field $\phi$ approaches fast (typically, exponentially fast) the asymptotic values $\phi_\pm$. Absolute minima of the potential correspond to the zeros of the superpotential, i.e. $U(\phi_\pm) = 0$. As it is clear from the field equations, the mass of fluctuations near the boundaries is given by

$$m_\pm^2 = (U'(\phi_\pm))^2. \tag{9.84}$$

Let us suppose for simplicity that

$$m_-^2 = m_+^2. \tag{9.85}$$

(If the masses are different, the consideration is a bit more involved technically.) Let us consider the set A, (9.77), of boundary conditions. The contribution of $\psi_+$

cancels 1/2 of the contribution of the bosonic field, and, consequently, we have noncompensated 1/2 of the contribution of a bosonic Robin mode with

$$\mathscr{S}_1 = -U'(\phi_-), \qquad \mathscr{S}_2 = U'(\phi_+). \tag{9.86}$$

Now, by using (9.30) and the relations between contribution from boundary conditions and bound states derived above, we calculate the vacuum energy associated with the boundaries:

$$\Delta H^{\text{bou}} = -\frac{1}{4\pi}(U'(\phi_+) - U'(\phi_-)), \tag{9.87}$$

which yields for the vacuum energy of the soliton

$$\Delta H^{\text{sol}} = \frac{1}{4\pi}(U'(\phi_+) - U'(\phi_-)). \tag{9.88}$$

In the particular case of the $\varphi^4$ kink (9.33) described by the superpotential (9.68) we have

$$\Delta H^{\text{kink}} = -\sqrt{\frac{\lambda}{2}}\frac{v_0}{\pi}. \tag{9.89}$$

As a consistency check one may verify that the set B, (9.78), of boundary conditions leads to the same expression for the vacuum energy.

The last problem which we have to discuss is what happens with the BPS bound and the supersymmetry in quantum theory? Let us remind that the solitons satisfy the BPS bound (the equality sign in (9.60)). This is a consequence of the Bogomolny equation (9.61), which, is equivalent to conservation of 1/2 of the supersymmetries. One has two consequences of quantum corrections: change of the mass of the kink which results in a violation of the BPS bound and a change of the central charge. It turns out that the two effects are consistent and compensate each other. Let us demonstrate this property in our approach.

Action (9.45) is invariant under SUSY transformations (9.46) up to boundary terms. It is not possible to introduce boundary terms in the action to compensate boundary terms generated by the SUSY. The reason is an algebraic one. Anticommutators of some of the SUSY transformations contain the translation operator on the right hand side. This fact implies an invariance of the theory under translations, which is inevitably broken by the presence of boundaries.

It is possible to preserve 1/2 of the supersymmetries in the presence of boundaries. Let us introduce a boundary functional

$$I^{\text{bou}} = \int dt\, Z \tag{9.90}$$

where $Z$ is defined in (9.66). One can easily check that the action $I + I^{\text{bou}}$, where the bulk action $I$ is given by (9.45), is invariant with respect to the $\epsilon_+$ transformations ($\epsilon_- = 0$) including the boundaries and without imposing any boundary conditions on the fields. Adding a new term to the action results in a contribution to the Hamiltonian, which reads now

$$H^{\text{tot}} = H - Z. \tag{9.91}$$

(Note that $Z$ does not contain time derivatives.) The canonical bracket of two supercharges (9.50), where the integration is restricted to a finite interval,

$$\{Q_-, Q_-\} = -2i H^{\text{tot}} \tag{9.92}$$

gives the total Hamiltonian without an extra term.

Let us make sure that the new boundary action is consistent with boundary conditions (9.77) and (9.78). Variation of the classical action $I + I^{\text{bou}}$ with respect to the fields produces equations of motion (9.47), (9.48) in the bulk plus the boundary term

$$\int_{\partial \mathscr{M}} dt\, n_x(\psi_- \delta\psi_+ + (\partial_x \varphi - U(\varphi))\delta\varphi). \tag{9.93}$$

Here $n_x$ is the $x$-component of the inward pointing unit normal to the boundary, i.e., $n_x = \pm 1$ at $x = \mp l$. This boundary term obviously vanishes for conditions (9.78). To show this for (9.77) one has to note that the condition for the bosonic fluctuations there is nothing else as a linearized version of the equation $\partial_x \varphi - U(\varphi) = 0$. Also in the part quadratic in fluctuations all boundary terms vanish, see Exercise 9.7.

We may conclude therefore that one-loop quantum correction $\Delta H^{\text{tot}}$ is the same as calculated above, i.e., $\Delta H^{\text{tot}} = 0$. Moreover, the initial bulk Hamiltonian $H$ may be identified with $H^{\text{sol}}$, and $H^{\text{bou}}$—with $-Z$. We see, that in this approach, quantum corrections to the mass of the soliton are compensated by quantum corrections to the central charge, and the BPS bound is saturated.

## 9.7 Literature Remarks

There exist many good books and review articles devoted specifically to the Casimir effect. Relatively recent sources are [45, 187, 188]. Therefore, the presentation in Sect. 9.2 was rather sketchy.

The first calculation of a force that appears between two parallel metal conducting plates was done by H. Casimir in 1948. The Casimir force is of the order of $10^{-3}$ N for the separation distance of 100 nm and the plates area of the order of one square meter. First experimental studies of the effect date back to 1958. Since that period the Casimir effect has been explored both theoretically and experimentally for a number of geometrical configurations. Its experimental verification is on the level of a few percents either for parallel plates of for interaction between a plate and a sphere. It should be noted that the sign of the Casimir force depends on the shape of the surfaces. For example, the Casimir energy of a single conducting sphere is positive, thus, the Casimir force is repulsive. The use of the zeta-regularization in the Casimir energy calculations was pioneered in [10, 38].

Some authors include in the notion of the Casimir effect all physical manifestation of the vacuum energy. The most important manifestation of vacuum fluctuations is related to their contributions to the cosmological constant $\Lambda$, see Eq. (7.68). There is a mounting number of cosmological data that the cosmological constant may constitute the so-called dark energy, an exotic form of matter which accelerates expansion of the Universe [204].

A modern development of the contour integration methods in calculations of spectral functions is reviewed in [170]. For a recent review on quantum corrections to masses of topological solitons one may consult [5].

Quantum corrections to the $\varphi^4$ kink in $1+1$ dimensions and the shift of kink mass (9.40) were first calculated by Dashen, Hasslacher and Neveu [74]. Other important early sources on quantum solitons are [106, 138, 211]. In our treatment of the vacuum energy in terms of the scattering data, contributions of the bound states, and the bosonic kink mass shift we follow [42] and [47]. The large mass subtraction scheme used to get (9.40) was discussed in some detail in [44, 45, 47].

The same mass shift (9.40) can be obtained in a framework of the renormalization procedure, when the divergences are removed by renormalizing the mass of the field $\varphi$, and demanding that the one loop effective potential has a minimum at $\varphi = v_0$, see [214]. One can show [47] that for all two-dimensional theories with a scalar background the condition above is equivalent to the large mass subtraction scheme used in Sect. 9.3. Moreover, the both methods are equivalent to the heat kernel subtraction scheme which is frequently used in calculations on a curved background [37] (when one drops the entire contribution of $a_2$ to the vacuum energy). A review of solitons is [211].

An introduction to supersymmetry can be found in [254, 256].

For a long time it was believed that due to the compensation of bosonic and fermionic contributions to the vacuum energy the mass shift of solitons in the presence of supersymmetry vanishes. It was demonstrated in [214] that carefully taking into account the renormalization effects one must obtain a non-zero vacuum energy of the kink in $1+1$ dimensions.

General aspects of supersymmetric solitons are reviewed in [231]. In (9.51) we used conventions of [158]. The appearance of a topologically non-trivial central charge, see Eq. (9.66) was first observed by Witten and Olive [259]. This effect is common for all SUSY models. After a non-zero mass shift of the supersymmetric kink was obtained [214] this effect has been interpreted as a new anomaly [143, 197, 232] (see [215]). In our treatment of the mass shift we follow [47]. The analysis of correction to the central charge is new, but it uses essentially the notion of "supersymmetry with boundary conditions" [32].

## 9.8 Exercises

**Exercise 9.1** Use two methods to compute the Casimir energy for a real massless scalar on a circle. The first method is to introduce regularized energy as

$$E_0(\epsilon) = \frac{1}{2}\sum_k \omega_k \, e^{-\epsilon\omega_k}, \tag{9.94}$$

where $\epsilon > 0$ is a regularization parameter. The second method is to use the Green's function of the fields and a point-splitting method described in Sect. 2.6, see (2.74), (12.276).

**Exercise 9.2** Consider quantum field theory on the so-called Einstein universe

$$ds^2 = -dt^2 + r^2 d\Omega_d^2, \tag{9.95}$$

where $d\Omega_d^2$ is the line element on a hypersphere $S^d$, $d = n - 1$, of a unit radius. Find the Casimir energy of a massless scalar field with conformal coupling (see the model (8.69)) on background (9.95). Consider for simplicity the case $n = 4$ and compare it with results of Exercise 9.1.

**Exercise 9.3** Find the vacuum energy on the Einstein universe for a Weyl spinor.

**Exercise 9.4** Check that action (9.45) is invariant under SUSY transformations (9.46).

**Exercise 9.5** Show that boundary conditions (9.77) and (9.78) are invariant under linearized SUSY transformations with $\epsilon_- = 0$.

**Exercise 9.6** By applying formula (9.88) to the anti-kink solution $\Phi_{\text{anti-kink}} = -\Phi_{\text{kink}}$ one obtains $\Delta H^{\text{anti-kink}} = -\Delta H^{\text{kink}}$. This result seems to contradict $P$-invariance of the theory (change of the sign of spatial coordinates). How can this contradiction be resolved?

**Exercise 9.7** By using the background field formalism, and choosing for simplicity a bosonic background field only, show that both bulk and boundary one-loop divergences in the theory described by the classical action $I + I^{\text{bou}}$ (see (9.45), (9.90)) are removed by a single renormalization of the superpotential.

# Chapter 10
# Open Strings and Born-Infeld Action

## 10.1 Open Strings in Background Gauge Fields

Spectral methods in the presence of non-trivial boundary conditions have many applications which go beyond the studies of the vacuum energy. In this Chapter we discuss boundary effects in a model of quantized extended one-dimensional objects, ‘strings’. Such objects or, more correctly, their supersymmetric extensions are considered in the string theory. They have very interesting properties and one of these properties is established here by applying the spectral theory.

We begin with necessary definitions. Our starting point is the so-called bosonic string. The ‘trajectory’ of a string, as distinct from the path of a particle, is a two-dimensional manifold $\mathscr{M}$ called a world-sheet. Thus, the bosonic string theory is two-dimensional. The coordinates on $\mathscr{M}$ are $\sigma$ (goes along the string) and $\tau$ (an analog of a proper time coordinate for a particle, see (1.24)). The manifold where the string propagates is called a target manifold. The coordinates of the target manifold are denoted as $X^\mu$ and the string embedding is described by equations $X^\mu = X^\mu(\sigma, \tau)$.

The strings with ‘free’ endpoints are called open strings. In this case the world-sheet manifold $\mathscr{M}$ has a boundary $\partial\mathscr{M}$. The following discussion is restricted to the case when the target space and, hence, the world-sheet have Euclidean signatures. The world-sheet metric is denoted as $h_{ab}(\sigma, \tau)$, $a, b = 1, 2$. The metric on the target space is $G_{\mu\nu}(X)$.

To determine equations for string ‘trajectories’ one considers $X^\mu(\sigma, \tau)$ as dynamical variables and, like in case of a particle action (1.24), introduces an action functional

$$I[X] = \frac{1}{4\pi\alpha'} \int_{\mathscr{M}} d^2x \sqrt{h}\, G_{\mu\nu} h^{ab} \partial_a X^\mu \partial_b X^\nu + \frac{1}{2\pi\alpha'} \int_{\partial\mathscr{M}} A_\mu(X)\, dX^\mu. \quad (10.1)$$

Here we use the convention $x^a = \{\sigma, \tau\}$, and $dX^\mu = \partial_\tau X^\mu\, d\tau$ along the boundary. The coupling $\alpha'$ is called the string tension. The first term in (10.1) is known as the Polyakov string action.

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Let us dwell on the symmetries of (10.1). First of all, this functional is invariant with respect to coordinate transformations in the target space as well as with respect to diffeomorphisms in the world-sheet. Additionally there are transformations of the boundary vector-function $A'_\mu(X) = A_\mu(X) + \partial_\mu \lambda(X)$ which do not change the action (10.1) since there is no boundary in the $\tau$ direction and corresponding integrations by parts do not produce any boundary terms. Thus, one can interpret $A_\mu(X)$ as an Abelian gauge field. It is known, that strings can interact with gauge fields only at their end points, and the second term in (10.1) describes such an interaction. Finally one can note that (10.1) has a conformal invariance related to local rescalings of the world-sheet metric $h_{ab}$.

From a geometrical view point equations following from the action (10.1) define the string world-sheet as a minimal surface in the Euclidean target space, i.e. the surface of a least area. From a different point of view, the coordinates $X^\mu$ are a set of two-dimensional fields while (10.1) is an action functional of some non-linear field model, so-called sigma model. To apply results obtained above in this book one has to reduce this model to a theory of free fields, i.e., to make a linearization over a background.

Let us suppose that the target space metric is constant Minkowski metric $G_{\mu\nu} = \delta_{\mu\nu}$ and consider fluctuations of the string, $X^\mu = \bar{X}^\mu + \sqrt{2\pi\alpha'}\,\xi^\mu$, which deviate it from some classical trajectory $\bar{X}^\mu$. The action for the fluctuation part $\xi^\mu$ follows from (10.1)

$$I_2[\bar{X},\xi] = \frac{1}{2}\int_{\mathscr{M}} d^2x\,\sqrt{h}\,\delta_{\mu\nu} h^{ab}\partial_a \xi^\mu \partial_b \xi^\nu + \frac{1}{2}\int_{\partial\mathscr{M}} d\tau \left(F_{\mu\nu}(\bar{X})\xi^\nu \partial_\tau \xi^\mu + \xi^\nu \xi^\rho \partial_\tau \bar{X}^\mu \partial_\nu F_{\rho\mu}(\bar{X})\right), \quad (10.2)$$

see Exercise 10.1. After integrating by parts in the volume term in (10.2) one obtains

$$I_2[\bar{X},\xi] = -\frac{1}{2}\int_{\mathscr{M}} d^2x\,\sqrt{h}\,\xi^\mu \delta_{\mu\nu} h^{ab}\nabla_a \nabla_b \xi^\nu + \frac{1}{2}\int_{\partial\mathscr{M}} d\tau \left(-\xi^\mu \delta_{\mu\nu}\partial_n \xi^\nu + F_{\mu\nu}(\bar{X})\xi^\nu \partial_\tau \xi^\mu + \xi^\nu \xi^\rho \partial_\tau \bar{X}^\mu\ \partial_\nu F_{\rho\mu}(\bar{X})\right). \quad (10.3)$$

If the boundary term in (10.3) is non-vanishing, it results in a delta-function-like potential with the support on $\partial\mathscr{M}$ in field equations for the string fluctuations $\xi^\mu$. To avoid this singular term one has to impose certain boundary conditions on $\xi^\mu$ which fix uniquely the spectrum of the fluctuations.

As we know, quantum effects may produce corrections to the boundary part of the effective action for $\xi^\mu$. Since the corrections depend on the boundary vector-potential $A_\mu$ they should be added to the boundary term in classical action (10.1). In this way the corrections may modify the boundary conditions on $\xi^\mu$ one started with. To ensure this does not happen and the theory is self-consistent one has to impose certain conditions on the vector-field $A_\mu$ itself. An interesting property of this simple model is that consistency conditions on $A_\mu$ are equivalent to equations in a classical non-linear electrodynamics of the Born-Infeld type.

## 10.2 Quantum Corrections for Oblique Boundary Conditions

The aim of this section is to derive a divergent part of the effective action for the string fluctuations in the presence of the boundary potential $A_\mu$ by using results of the spectral theory.

First, let us discuss boundary conditions on $\xi^\mu$ which eliminate the boundary term in (10.3). It is clear that one can impose either Dirichlet conditions

$$\xi^\mu|_{\partial\mathscr{M}} = 0, \tag{10.4}$$

or a kind of generalized Neumann conditions

$$\left(-\delta_{\mu\nu}\partial_n + \frac{1}{2}(F_{\mu\nu}\partial_\tau + \partial_\tau F_{\mu\nu}) + (\partial_\tau \bar{X}^\rho)\cdot\frac{1}{2}(\partial_\nu F_{\mu\rho} + \partial_\mu F_{\nu\rho})\right)\xi^\mu|_{\partial\mathscr{M}} = 0. \tag{10.5}$$

Equations (10.5) are presented in a manifestly symmetric form, for the reasons which are explained later. Such a symmetrization is always possible since it does not affect the quadratic form of the action.

It is allowed to mix up boundary conditions (10.4) and (10.5), i.e. to assume that some components of $\xi^\mu$ satisfy (10.5) while the other components obey (10.4). Conditions (10.5) are genuine open string conditions. The Dirichlet condition (10.4) means that some of the string endpoints are confined to a subsurface in the target space which is called a Dirichlet brane or simply a D-brane. Conditions (10.5) belong to the class of the oblique boundary conditions (3.46).

Let us study now the boundary value problem for a Laplacian with the boundary conditions

$$\mathscr{B}\varphi = \left(\nabla_n + \frac{1}{2}(\Gamma^i\nabla_i + \nabla_i\Gamma^i) + \mathscr{S}\right)\varphi|_{\partial\mathscr{M}} = 0. \tag{10.6}$$

This condition is practically the same as (3.46) except that in (10.6) we use a symmetric combination of $\nabla_i$ in $\Gamma^i$ in order to simplify the hermiticity analysis. The difference between (3.46) and (10.6) is just a shift of $\mathscr{S}$.

By comparing (10.6) with (10.5) one can see that

$$\Gamma_{\nu\mu} = -F_{\mu\nu}, \tag{10.7}$$

$$\mathscr{S}_{\nu\mu} = -(\partial_\tau \bar{X}^\rho)\cdot\frac{1}{2}(\partial_\nu F_{\mu\rho} + \partial_\mu F_{\nu\rho}). \tag{10.8}$$

The connection for this system is trivial. Since the boundary is one-dimensional we do not write a world-sheet vector index for $\Gamma$. Both $\Gamma$ and $\mathscr{S}$ are matrices with target-space indices. Note, that $\Gamma$ is antisymmetric, while $\mathscr{S}$ is symmetric. This is, in fact, a general feature. The Laplacian is formally selfadjoint with boundary conditions (10.5) provided that $\Gamma^i$ and $\mathscr{S}$ are anti-Hermitian and Hermitian matrices, respectively. This statements is an easy extension of Exercise 3.1. If one adopts different conventions for the Euclidean theory so that an extra $i$ appears in front of $A_\mu$, the hermiticity properties of $\mathscr{S}$ and $\Gamma$ are reversed, and the Laplacian is no longer selfadjoint.

Let us discuss leading coefficients of the heat kernel expansion. We leave without a proof that asymptotic expansion (4.9) indeed takes place for moderate $\Gamma$, and the coefficients are locally computable as in Sect. 4.5. However, when we try to write down general expressions for the heat kernel coefficients we immediately meet a difficulty. The canonical mass dimension of $\Gamma^i$ is zero, and therefore, $\Gamma^i$ may appear in the heat kernel coefficients in *arbitrary power*. Besides, in general $\Gamma^i$ do not commute with each other, so we have some ordering ambiguities. To make the situation easier we suppose that all $\Gamma^i$ commute. This assumption does not impose any restrictions on the field configurations in the case of strings since there is just one $\Gamma$. The coefficient $a_0$, which has volume contributions only, remains as before (4.56). For next two coefficients one can write

$$a_1(L) = (4\pi)^{-(n-1)/2} \int_{\partial\mathscr{M}} d^{n-1}x \sqrt{h}\, \mathrm{tr}(\gamma(\Gamma)), \tag{10.9}$$

$$a_2(L) = (4\pi)^{-n/2} \frac{1}{6}\Bigg[\int_{\mathscr{M}} d^n x \sqrt{g}\, \mathrm{tr}(f(6E+R)) + \int_{\partial\mathscr{M}} d^{n-1}x \sqrt{h}\, \mathrm{tr}\big(b_0(\Gamma)K^j_j + b_2(\Gamma)\mathscr{S} + \sigma(\Gamma)K_{ij}\Gamma^i\Gamma^j\big)\Bigg]. \tag{10.10}$$

To avoid a confusion we should note that in these expressions we follow conventions of Sect. 4.5 for the case of $n$-dimensional base manifold $\mathscr{M}$. In (10.9), (10.9) we introduced $\gamma$, $b_0$, $b_2$ and $\sigma$ as arbitrary functions of $\Gamma^2$. A somewhat strange nomenclature is consistent with references discussed in Sect. 10.4. The calculations of $\gamma$, $b_0$, $b_2$ and $\sigma$ are rather involved but still doable with the help of a rather smart extension of standard methods [19–21, 94, 186]. The result reads

$$\begin{aligned} \gamma &= \frac{1}{4}\left[\frac{2}{\sqrt{1+\Gamma^2}} - 1\right], \\ b_0 &= 6\left[\frac{1}{1+\Gamma^2} - \frac{1}{\sqrt{-\Gamma^2}}\,\mathrm{artanh}\big(\sqrt{-\Gamma^2}\big)\right] + 2, \\ b_2 &= \frac{12}{1+\Gamma^2}, \\ \sigma &= \frac{1}{\Gamma^2}(2 - b_0). \end{aligned} \tag{10.11}$$

In the limit $\Gamma \to 0$ one reproduces the heat kernel coefficients for Neumann boundary conditions (4.74) and (4.75).

As we have already mentioned, $\Gamma^i$ are typically anti-Hermitian. Therefore, $\Gamma^2 \le 0$. As $\Gamma^2$ approaches $-1$ the heat kernel coefficients blow up due to the loss of strong ellipticity of the boundary value problem. This phenomenon has been briefly discussed in Sect. 3.2, see (3.37)–(3.41). In the context of the string theory the divergence points out to a critical value of the filed strength $F_{\mu\nu}$.

Now we are ready to evaluate divergent part of the one-loop effective action for open strings

$$W[A] = \frac{1}{2}\ln\det\Delta[A],$$

where $\Delta$ is the Laplace operator on the string-world sheet $\mathscr{M}$ for boundary conditions (10.5). In zeta-function regularization (5.51)

$$W_{\rm div}[A] = -\frac{1}{2s} a_2(\Delta). \tag{10.12}$$

Since the target space is flat the string world-sheet can be chosen as a flat minimal surface. This is possible at least when $\partial\mathscr{M}$ is a plane, $K_{ij} = 0$. If these conditions are assumed one gets from (10.7), (10.8), (10.10), (10.11), and (10.12) the following result:

$$W_{\rm div}[A] = \frac{1}{4\pi s} \int_{\partial\mathscr{M}} d\tau\, \partial_\tau \bar{X}^\rho \cdot (\partial_\mu F_{\nu\rho} + \partial_\nu F_{\mu\rho})[1 + F^2]^{-1\mu\nu}. \tag{10.13}$$

We do not derive Eqs. (10.11) which are used to establish the boundary divergences in the effective action. This can be found in the references given above. To justify (10.13) one can use an alternative approach based on the Green's functions for open strings, see Exercises 10.3–10.5.

## 10.3 The Born-Infeld Action and Noncommutative Coordinates

Correction (10.13) looks similar to the boundary term in classical action (10.1). This motivates us to write

$$W_{\rm div}[A] = \frac{1}{s} \frac{1}{2\pi\alpha'} \int_{\partial\mathscr{M}} d\tau\, \partial_\tau \bar{X}^\rho \cdot \beta^A_\rho, \tag{10.14}$$

$$\beta^A_\rho \equiv \frac{1}{2}\alpha' (\partial_\mu F_{\nu\rho} + \partial_\nu F_{\mu\rho})[1 + F^2]^{-1\mu\nu}. \tag{10.15}$$

As was pointed out in Sect. 10.1 the theory is consistent if

$$\beta^A_\mu = 0. \tag{10.16}$$

This condition also ensures conformal invariance in the string theory, though we shall not go into details of this interpretation. Note, that $\beta^A_\mu$ depends on $A_\mu$ and on the derivatives of $A_\mu$ with respect to the target space coordinates. A remarkable fact is that condition (10.16) is equivalent to the equations which can be derived from the following action on the target space:

$$I_{\rm BI} = \int d^N X \sqrt{\det(\delta_{\mu\nu} + i F_{\mu\nu})}. \tag{10.17}$$

Functional (10.17) is called the Born-Infeld action (or the Dirac-Born-Infeld action). $N$ is the number of target space dimensions. The factor $i$ appeared in Euclidean space due to our conventions.

In the weak field approximation, when $F_{\mu\nu}$ is small, (10.17) can be expanded in a power series. The first non-trivial term, which is an integral of $F_{\mu\nu}F^{\mu\nu}$, has the standard Maxwell form. The Born-Infeld action, therefore, describes a non-linear extension of the standard Maxwell electrodynamics.

Another remarkable result of the theory of open strings is a noncommutativity of the coordinates of the string endpoints. Let us evaluate the commutator $[X^\mu(\tau), X^\nu(\tau)]$ when both operators are taken at the boundary for constant $\Gamma$. To this end we need the Green's function of 'fields' $X^\mu$ taken on the boundary. The Green's function is computed in Exercise 10.2, see (10.23). Its boundary value up to an inessential constant part reads

$$G(\tau, \tau') = \mathscr{G} \ln(\tau - \tau')^2 + \frac{1}{2}\frac{\Gamma}{1+\Gamma^2} \operatorname{sign}(\tau - \tau'), \tag{10.18}$$

where $\mathscr{G}$ is a symmetric matrix whose precise form plays no role. One can argue, that this Green's function is exactly the vacuum expectation value

$$G(\tau, \tau') = \langle X(\tau) X(\tau') \rangle \tag{10.19}$$

and that the commutator is recovered in the time-ordered limit

$$[X^\mu(\tau), X^\nu(\tau)] = \lim_{\epsilon \to 0} \big[ G^{\mu\nu}(\tau, \tau - \epsilon) - G^{\mu\nu}(\tau, \tau + \epsilon) \big] \tag{10.20}$$

yielding

$$[X^\mu, X^\nu] = i \delta^{\mu\nu} \theta, \tag{10.21}$$

where

$$\theta = \frac{\Gamma}{1 - \Gamma^2}. \tag{10.22}$$

The factor $i$ appears in (10.21) due to continuation to the Minkowski signature space accompanied by $\Gamma \to i\Gamma$, which we performed at the last step.

We conclude that the coordinates of the string endpoints do not commute, and, hence, an effective field theory on a D-brane has to be a theory constructed on a noncommutative space. Such theories are discussed in the next Chapter.

## 10.4 Literature Remarks

String theory belongs to one of the most exciting areas of modern theoretical physics. String is seen as an extended object at very high energies only. The low-energy limit of strings is defined by the condition that certain string beta-functions vanish. These beta-functions can be calculated by the methods of quantum field theory in external fields. An extensive introduction to string theory can be found in the textbooks [144, 208].

The metric $G_{\mu\nu}$ and the boundary vector field $A_\mu$ are fields on the target space. From the world-sheet standpoint they can be also viewed as sets of couplings. Each of these fields can be expanded in a Taylor series, and the coefficients in front of the powers of $X^\mu$ play a role of independent coupling constants. Therefore, the string action (10.1) describes a two-dimensional field theory with an infinite number of couplings. Since $A_\mu(X)$ plays the role of couplings correction (10.14) represents counterterms to these couplings. In this sense $\beta_\rho^A$ introduced in (10.15) are related to beta-functions for the couplings $A_\mu$.

The Born-Infeld action was derived from open strings by Fradkin and Tseytlin [112, 239]. The derivation based on the string beta-functions was presented in the works [1, 56], from which we borrowed some of the material. The Dirichlet conditions on some of the string coordinates (the D-branes) were discussed in this context in [73, 178]. Latter on the Dirichlet branes became important due to the seminal work by Polchinski [207].

We have not fixed the dimensionality of the target manifold $N$. The bosonic string theory is consistent provided that the target space has 26 dimensions. In many sources an extra multiplier of $i$ appears in front of $A_\mu$ in (10.1) meaning that different conventions for the continuation to the Euclidean space are used. Our conventions ensure that the fluctuation operator is selfadjoint.

Oblique boundary conditions have been studied in mathematics since a long time, see [147]. The problem of calculating the heat trace asymptotics for these boundary conditions was first addressed in [186], and the results of this work were later extended and improved by Avramidi and Esposito [19–21] and by Dowker and Kirsten [94] (the former authors discuss also the strong ellipticity in the context of these boundary conditions). The heat kernel calculations were applied to string theory in [176, 203].

Our derivation of noncommutativity of the coordinates of string end points follows the analysis of [223, 230].

## 10.5 Exercises

**Exercise 10.1** Calculate second-order terms in the background field expansion of open string action (10.1).

**Exercise 10.2** Let $\mathscr{M} = \mathbb{R} \times \mathbb{R}_+$, so that the coordinates on $\mathscr{M}$ have the following ranges: $\tau \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+$. Let $\Delta$ be a Laplace operator on $\mathscr{M}$ subject to boundary conditions (10.6) with a constant $\Gamma$ and $\mathscr{S} = 0$. Prove that the Green's function $G = \Delta^{-1}$ has the following form:

$$G(x,x') = -\frac{1}{4\pi}\left[\ln|z-z'|^2 + \frac{1+i\Gamma}{1-i\Gamma}\ln(z-\bar{z}') + \frac{1-i\Gamma}{1+i\Gamma}\ln(\bar{z}-z')\right], \quad (10.23)$$

where $z = \tau + i\sigma$, $z' = \tau' + i\sigma'$. Since the imaginary part of $z-\bar{z}'$ is always positive, and the imaginary part of $\bar{z}-z'$ is always negative, it is convenient to put the branch cut of the logarithm along $\mathbb{R}_-$.

**Exercise 10.3** Let $\mathscr{M} = \mathbb{R} \times \mathbb{R}_+$ and let $G_0(z,z')$ be the Green's function of the free scalar Laplacian $\Delta = -\partial^2$ satisfying Neumann boundary condition $\partial_\sigma G_0(z,z')|_{\sigma=0} = 0$. Demonstrate that the two-point function $G_S(z,z')$ obeying the Dyson equation

$$G_S(z,z') = G_0(z,z') + \int d\tau''\, G_0(z;\tau'',0)S(\tau'')G_S(\tau'',0;z') \quad (10.24)$$

is the Green's function of the same operator satisfying the Robin boundary condition

$$(\partial_\sigma + S(\tau))G_S(z, z')|_{\sigma=0} = 0. \tag{10.25}$$

Solve (10.24) perturbatively.

The method which allows one to construct perturbative Green's functions satisfying various boundary conditions from free (or simpler) Green's functions is called the multiple reflection expansion [22, 23, 153]. The example considered here is taken from [46, 176].

**Exercise 10.4** Consider an effective action $W$ for a theory with oblique boundary conditions (10.6). The dependence of $W$ on the boundary function $\mathscr{S}$ can be established perturbatively if $\mathscr{S}$ is sufficiently small. Use the multiple reflection expansion discussed in Exercise 10.3 to show that the first term in the expansion of the action in $\mathscr{S}$ is

$$W_1(\mathscr{S}) \propto \int_{\partial\mathscr{M}} d\tau \ \mathrm{tr}(\mathscr{S}(\tau)G(\tau, 0; \tau', 0)), \tag{10.26}$$

where $G$ is the Green's function (10.23) and $|\tau - \tau'|$ is assumed to be a small regularization parameter.

**Exercise 10.5** Use results of Exercise 10.4 and Eq. (10.26) to derive divergent part (10.13) of the string effective action.

**Exercise 10.6** Prove that the equations of motion for the Born-Infeld action (10.17) yield conditions (10.16).

# Chapter 11
# Noncommutative Geometry and Field Theory

## 11.1 Motivations for Noncommutative Geometry

In this last Chapter we go from a traditional material to new frontiers where the methods of spectral geometry can be applied. We consider noncommutative theories which are a beautiful example of how physics and mathematics have a mutual influence.

When studying open strings we have encountered a space with noncommuting coordinates, see (10.21). Such spaces are called noncommutative (NC). Open strings suggest an example of a real noncommutative plane with the following commutator of the coordinates:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{11.1}$$

where $\theta^{\mu\nu}$, in general, is a constant real anti-symmetric tensor.

There are several reasons to formulate quantum theories on the noncommutative plane or on its generalizations. The key point is that quantum gravity effects prevent coordinates from being simultaneously measurable with arbitrary high precision at very small scales. In accelerator experiments a large energy of colliding particles is concentrated in a small region. Increasing the energy cannot be unlimited because at a certain value the region collapses to form a microscopic black hole. Some information may be lost inside the black hole horizon and this fact implies that quantum-mechanical operators corresponding to the coordinates do not commute between themselves.

The noncommutativity is not necessarily a property of high energy phenomena. It may exist in physical systems which are not related to quantum gravity, for a example in condensed matter under description of so-called planar electrons in an external magnetic field. Such systems are interesting because of applications to the quantum Hall effect.

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## 11.2 The Star Product

To construct models on noncommutative spaces one needs to revise a number of notions from Chap. 1. The first step is to find a suitable realization of commutation relation (11.1). A straightforward approach would be to represent (11.1) by operator-valued coordinates $x^\mu$. However, this way is not convenient because in such approach even a classical field $\varphi(x)$ on the NC plane becomes an operator-valued function.

A more simple option is to replace the usual commutative product of functions by a new noncommutative *star* product. Let $f$ and $g$ be two smooth functions from the Schwartz space on $\mathbb{R}^n$, i.e. functions which vanish as $|x^\mu| \to \infty$ together with all their derivatives. On the Schwartz space the star product can be defined by the Moyal formula

$$(f \star g)(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x \partial_\nu^y\right) f(x) g(y)\Big|_{y=x}. \qquad (11.2)$$

The exponential in (11.2) should be understood as a formal expansion in the noncommutativity parameter $\theta^{\mu\nu}$, though a strict convergence of this expansion is rather problematic, in general. With some efforts, however, the formula (11.2) can be replaced by a convergent expression, the so-called Rieffel formula. We skip further details. The star product is an essentially non-local operation. The usual product is recovered from (11.2) in the limit $\theta^{\mu\nu} \to 0$.

With some precautions, since $x^\nu$ are not from the Schwartz space, the Moyal formula can be applied to coordinates themselves. In mathematical language this corresponds to considering the multiplier algebra [140]. In particular, one can check that

$$x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}. \qquad (11.3)$$

If the left hand side of (11.3) is identified with a commutator one gets a required realization of commutation relation (11.1). The NC plane with the Moyal product is called the Moyal plane.

One can check several basic properties of star product (11.2). The product is associative and it is Hermitian,

$$(f \star g)^* = g^* \star f^*, \qquad (11.4)$$

with respect to the complex conjugation operation. The product is closed in the sense that

$$\int d^n x\, f \star g = \int d^n x\, f \cdot g. \qquad (11.5)$$

The star multiplication by a plane wave is a composition of the ordinary product and a shift

$$f \star e^{ikx} = f(x^\mu - \theta^{\mu\nu} k_\nu/2) e^{ikx}, \qquad e^{ikx} \star f = f(x^\mu + \theta^{\mu\nu} k_\nu/2) e^{ikx}. \qquad (11.6)$$

The product of two plane waves looks very simple,

$$e^{ikx} \star e^{iqx} = e^{i(k+q)x} e^{-\frac{i}{2} k\wedge q}, \qquad k \wedge q \equiv k_\mu \theta^{\mu\nu} q_\nu. \tag{11.7}$$

Since left and right actions by multiplications of the algebra of functions on itself are different it is convenient to introduce two multiplication operators,

$$L(f_1) f_2 \equiv f_1 \star f_2, \qquad R(f_1) f_2 \equiv f_2 \star f_1, \tag{11.8}$$

the left and the right ones, respectively.

The star product is a convenient tool to formulate a field theory on the NC plane. One can simply take a field action on a commutative manifold and replace everywhere the usual products by the star products. For example, an NC version of Euclidean $\varphi^4$ constructed in this way is

$$I[\varphi] = \frac{1}{2} \int d^n x \left( (\partial_\mu \varphi)^2 + m^2 \varphi^2 + \frac{\lambda}{12} \varphi \star \varphi \star \varphi \star \varphi \right). \tag{11.9}$$

The quadratic terms do not contain star multiplication due to (11.5). A non-local nature of the noncommutative field theory associated with the star operation appears in the interacting terms.

Such a straightforward formulation of the noncommutative theory is not unique but we shall accept it for further discussion.

## 11.3 The Heat Trace of Operators on the Moyal Plane

Let us introduce some basic spectral functions of operators on the Moyal plane and discuss their main properties.

To determine a possible structure of the operators we use model (11.9) as a guide. The decomposition $\varphi = \phi + \chi$ of the field in (11.9) into a background part $\phi$ and a fluctuation $\chi$ yields in the second order the action

$$I_2[\chi, \phi] = \frac{1}{2} \int d^n x\, \chi P(\phi) \chi. \tag{11.10}$$

Here $P$ is the following operator:

$$P = -\partial_\mu^2 + m^2 + \frac{\lambda}{6} (R(\phi \star \phi) + L(\phi \star \phi) + L(\phi) R(\phi)), \tag{11.11}$$

which is a generalization of a scalar Laplacian where a potential term is replaced by the terms with left and right Moyal multiplications. If a connection were present it might have contained left and right parts, $L(\lambda_\mu)$ and $R(\rho_\mu)$, respectively. Thus, a fairly general Laplacian on the Moyal plane is

$$P = -(\nabla_\mu \nabla^\mu + \hat{E}), \tag{11.12}$$

$$\begin{aligned} \nabla_\mu &= \partial_\mu + R(\rho_\mu) + L(\lambda_\mu), \\ \hat{E} &= L(l_1) + R(r_1) + L(l_2) R(r_2), \end{aligned} \tag{11.13}$$

where functions $l_k$, $r_k$, and components $\rho_\mu$, $\lambda_\mu$ of vector fields belong to the Schwartz space. The metric in (11.12) is flat.

Let us illustrate, in a quite unrigorous way, how effects of noncommutativity appear in the spectral theory. As an example we consider the unsmeared heat trace

$$K(P;t) = \mathrm{Tr}(e^{-tP}) \tag{11.14}$$

and estimate its behavior at small $t$ for the class of operators (11.12).

As follows from (11.6) and (11.7) the Moyal multiplication is a combination of the usual multiplication and a shift. The shift is an operator of a unit norm. As a consequence, the Moyal multiplication by a smooth function is a bounded operator, see Sect. 3.3. One may expect, therefore, that it is allowed to isolate in $e^{-tP}$ the "main part", $e^{-t\Delta}$ with $\Delta = -\partial_\mu \partial^\mu$, and expand the rest of the exponential in the power series of $\nabla$ and $\hat{E}$, as has been done in the commutative case, cf. Eqs. (4.44)–(4.48).

It is enough to understand how to deal with a typical term in this expansion which has the structure

$$T(l,r) = \mathrm{Tr}(L(l)R(r)e^{-t\Delta}), \tag{11.15}$$

where $l$ and $r$ are some smooth functions of the background fields appearing in the operator $P$. This quantity is a heat trace for the free Laplace operator with two smearing functions, one acting from the left, and one from the right, i.e.

$$T(l,r) \equiv K(L(l)R(r), \Delta, t). \tag{11.16}$$

The trace in (11.15) can be calculated by sandwiching the expression between plane waves, $e^{-ikx}$ and $e^{ikx}$, and integrating over $k$. It is also convenient to expand $l(x)$ and $r(x)$ in the Fourier integrals

$$\begin{aligned} r(x) &= \frac{1}{(2\pi)^{n/2}} \int d^n q\, \tilde{r}(q) e^{iqx}, \\ l(x) &= \frac{1}{(2\pi)^{n/2}} \int d^n q'\, \tilde{l}(q') e^{iq'x}. \end{aligned} \tag{11.17}$$

If there are compact dimensions, one has to use sums instead of the integrals. One obtains

$$T(l,r) = \int d^n x \int \frac{d^n k}{(2\pi)^n} e^{-tk^2} \langle L(l)R(r)\rangle_k, \tag{11.18}$$

where

$$\langle L(l)R(r)\rangle_k \equiv e^{-ikx} \star l(x) \star e^{ikx} \star r(x). \tag{11.19}$$

That is, the operator $R(r)L(l)$ acts on $e^{ikx}$, and the result is multiplied with $e^{-ikx}$ from the left. One finds

$$\langle L(l)R(r)\rangle_k = \frac{1}{(2\pi)^n} \int d^n q\, d^n q'\, \tilde{r}(q)\tilde{l}(q') e^{i(q+q')x} e^{\frac{i}{2}k\wedge(q-q')} e^{-\frac{i}{2}(q'-k)\wedge(q+k)}. \tag{11.20}$$

One can then integrate over $x$, get a delta-function $\delta(q+q')$, and integrate with its help over $q'$. The result is

$$T(l,r) = \int \frac{d^n k\, d^n q}{(2\pi)^n} e^{-tk^2} \tilde{l}(-q)\tilde{r}(q) e^{-ik\wedge q}. \tag{11.21}$$

To get some experience of working with such expressions one may consider the case when either $r(x)$ or $l(x)$ is a constant. In the terminology used in perturbation expansions in an NC quantum field theory, this is the case of so-called *planar* diagrams. One easily obtains the expression

$$T(l,1) = \frac{1}{(4\pi t)^{n/2}} \int d^n x\, l(x), \tag{11.22}$$

which coincides with the one for the commutative case. The same result can be obtained on the torus. In (11.21) no asymptotic expansion is assumed yet.

Note that constant functions are not from the Schwartz space since they do not decay at the infinity. The asymptotic expansions for Schwartz class smearing functions may look quite differently. Moreover, for non-constant functions (the so-called non-planar case) the result depends on weather the space is compact or not.

To illustrate these properties we choose $\mathscr{M} = \mathbb{R}^n$ and perform the integration over $k$ by completing the square in the exponential

$$T(l,r) = \int \frac{d^n q}{(4\pi t)^{n/2}} \tilde{l}(-q)\tilde{r}(q) \exp\left( -\frac{1}{4t} \theta^{\mu\mu'} \theta^{\nu}{}_{\mu'} q_\mu q_\nu \right). \tag{11.23}$$

In the commutative limit, $\theta = 0$, Eq. (11.23) reproduces the heat trace of the free Laplace operator smeared with the function $f(x) = l(x)r(x)$.

If $\theta^{\mu\nu}$ is non-degenerate one may note that contributions to the integral in (11.23) are exponentially suppressed for momenta which lie in the region $|\theta^{\mu\nu} q_\nu| > \sqrt{t}$. Therefore, in the limit $t \to 0$ one can decompose $\tilde{l}(-q)$ and $\tilde{r}(q)$ near $q = 0$ in Taylor series. Leaving only the leading terms in the series one obtains

$$T(l,r) = (\det\theta)^{-1} [\tilde{l}(0)\tilde{r}(0) + \mathscr{O}(t)], \tag{11.24}$$

or, in the coordinate representation,

$$T(l,r) = (\det\theta)^{-1} \frac{1}{(2\pi)^n} \int d^n x\, l(x) \int d^n y\, r(y) + \mathscr{O}(t). \tag{11.25}$$

Formulas (11.22) and (11.25) allow one to make two important conclusions regarding the heat trace asymptotics for a non-degenerate $\theta$.

First, if only left or only right Moyal multiplications appear in the expansion of the operator exponent in the heat trace the corresponding contributions to the asymptotics have the same structure as in the commutative case. The differences are in numerical coefficients and in changing usual products of fields to the star products.

Second, if the both types of multiplications appear in the same monomial, the corresponding contributions are essentially non-local and have no smooth commutative limit. Fortunately, such terms are $\mathscr{O}(t^0)$. Moreover, if we are interested in terms containing the fields from the operator $P$ rather than the smearing functions only, additional powers of $t$ appear, making the total power of the proper time positive. Hence, if the results of commutative quantum theories are extended to NC QFT's one may conclude on the base of Eq. (5.74) that non-planar terms do not bring new divergences at one loop.

An explicit example of the heat kernel calculations on a Moyal plane is considered in Exercise 11.2.

## 11.4 Quantization of Noncommutative Solitons

Let us consider the spectrum of quantum fluctuations above NC solitons in $1+1$ dimensions. As a model we take an NC version of action (9.32)

$$I[\varphi] = -\frac{1}{2}\int d^2x \left( (\partial_\mu \varphi)^2 + \frac{\tilde{\lambda}}{2}(v_0^2 - \varphi \star \varphi)^2 \right). \tag{11.26}$$

We put twiddle over $\lambda$ to avoid confusions with another coupling constant used in (11.9). The relation between couplings reads $m^2 = -\tilde{\lambda} v_0^2$, $\lambda = 6\tilde{\lambda}$.

On *static* solutions of the equations of motion following from (11.26) the star multiplication in two dimensions coincides with the usual multiplication and the noncommutativity plays no role. Therefore, the commutative kink (9.33) is still a solution. However, the spectrum of fluctuations is deformed. The fluctuations are described by the linearized equation

$$[\partial_0^2 - \partial_1^2 - \tilde{\lambda} v_0^2 + \tilde{\lambda}(L(\phi^2) + R(\phi^2) + L(\phi)R(\phi))]\chi = 0 \tag{11.27}$$

with $\phi \equiv \phi_{\text{kink}}$. Since the kink is static, we can look for the solutions of (11.27) in the form $\chi = e^{i\omega t}\eta_\omega(x)$. In two dimensions any skew-symmetric matrix can be represented as

$$\theta^{\mu\nu} = 2\Theta \epsilon^{\mu\nu}, \tag{11.28}$$

where $\Theta$ is a number. Hence, we obtain the following equation for $\eta$:

$$[\omega^2 + \partial_x^2 + \tilde{\lambda} v_0^2 - \tilde{\lambda}(\phi_+^2 + \phi_-^2 + \phi_+\phi_-)]\eta_\omega(x) = 0, \tag{11.29}$$

where $\phi_\pm(x) = \phi(x_\pm)$, $x_\pm = x \pm \Theta\omega$. For the sake of simplicity let us put

$$\tilde{\lambda} v_0^2 = 2. \tag{11.30}$$

One can then rewrite (11.29) as

$$L(\omega)\eta_\omega \equiv [-\partial_x^2 + M^2 + V(x;\omega)]\eta_\omega = \omega^2 \eta_\omega, \tag{11.31}$$

where the constant $M^2 = 4$ is selected in such a way that the potential

$$V(x;\omega) = 2(\tanh^2(x_+) + \tanh^2(x_-) + \tanh(x_+)\tanh(x_-) - 3) \tag{11.32}$$

vanishes for $x \to \pm\infty$. An example of $V(x;\omega)$ for $\Theta\omega = 4$ is shown on Fig. 11.1. We see that this potential has a form of a well with a frequency-dependent effec-

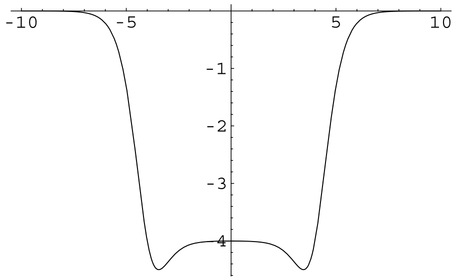


**Fig. 11.1** The potential $V(x;\omega)$ for $\Theta\omega = 4$ as a function of $x$

tive width of order $\Theta\omega$. For large $\Theta\omega$ the potential looks practically as a square well. One can derive an intriguing property of the corresponding non-linear spectral problem: at large $\Theta$ the number of bound states grows linearly with $\Theta$ [249].

Let us define a spectral density $\rho(\omega)$ for non-linear spectral problem (11.31) by requiring that $\rho(\omega)d\omega$ is the number of solutions for (11.31) with the eigenvalues between $\omega$ and $\omega + d\omega$. In the case of a discrete spectrum the spectral density is a sum of delta-functions, see Sect. 5.37. With this spectral density, we can define a regularized vacuum energy

$$E(s) = \frac{1}{2}\int d\omega\, \rho(\omega)(\omega^2)^{\frac{1}{2}-s} \tag{11.33}$$

and a pseudo-trace

$$K(t) = \int d\omega\, \rho(\omega)e^{-t\omega^2}, \tag{11.34}$$

see (6.16). Note that (11.33) coincides with the definition of the vacuum energy in commutative theories. We just take this fact as granted without any proof.

As an example of quantum calculations in NC theories let us evaluate the divergent part of vacuum energy (11.33). We shall use the techniques developed in Chap. 6.

Let us suppose that there is an asymptotic expansion (6.17) for (11.34). (A more careful analysis shows that this is indeed true.) The functions $E(s)$ and $K(t)$ are related through the Mellin transform. Particular form and the origin of the spectral density $\rho(\omega)$ is not essential as long as the assumption regarding the asymptotic properties of $K(t)$ holds. The pole part of the vacuum energy is given, therefore, by relation (9.3),

$$E^{\text{pole}} = \frac{1}{4\sqrt{\pi}}\frac{1}{s}a_2. \tag{11.35}$$

To calculate $a_2$, according to Chap. 6, one has to consider the heat kernel for an auxiliary spectral problem

$$K(L(\omega);t) = \text{Tr}(e^{-tL(\omega)} - e^{-t(-\partial_x^2+M^2)}). \tag{11.36}$$

Here an "empty space" contribution was subtracted for convenience. The spectral problem for $L(\omega)$ with a fixed $\omega$ is a very simple problem with a fixed potential. Obviously, there is an asymptotic expansion

$$K(L(\omega);t) \simeq \sum_{n=1}^{\infty} t^{n-1/2} a_{2n}(\omega), \tag{11.37}$$

$$a_2(\omega) = -(4\pi)^{-1/2} \int dx\, V(x;\omega), \tag{11.38}$$

$$a_4(\omega) = (4\pi)^{-1/2} \int dx \left[ \frac{1}{2} V(x;\omega)^2 + M^2 V(x;\omega) \right]. \tag{11.39}$$

By using (11.32) one obtains

$$a_2(\omega) = \frac{4}{\sqrt{\pi}} (\Theta\omega \coth(2\Theta\omega) + 1). \tag{11.40}$$

For large $\omega$ we have up to exponentially small terms (e.s.t.)

$$a_2(\omega) = \omega a_{1,1} + a_{0,1} + \text{e.s.t.} \tag{11.41}$$

$$a_{1,1} = \frac{4\Theta}{\sqrt{\pi}}, \qquad a_{0,1} = \frac{4}{\sqrt{\pi}}, \tag{11.42}$$

see (6.19). For a future use we note that

$$a_{0,1} = -\frac{\tilde{\lambda}}{\sqrt{\pi}} \int dx \left(\phi^2 - v_0^2\right). \tag{11.43}$$

Here one can use an explicit form of the kink solution and restore the $v_0$-dependence with the help dimensional arguments.

Other heat kernel coefficients have a similar behavior

$$a_{2p}(\omega) = \omega a_{1,p} + a_{0,p} + \text{e.s.t.} \tag{11.44}$$

This spectral problem is an *asymptotically polynomial NLSP* in the terminology of Chap. 6. One can use (6.20) to obtain

$$a_2 = a_{0,1}. \tag{11.45}$$

This formula together with Eq. (11.35) defines the divergent part of the vacuum energy. By using (11.43) and comparing it to the classical action (11.26) one immediately concludes that the divergence may be canceled by a renormalization of $v_0^2$, which is the same as the mass renormalization.

One can also prove that the pole term is $2/3$ of the corresponding value in the commutative case. The easiest way to see this is to repeat the calculations starting with (11.40) directly at $\Theta = 0$. This yields $a_2(\omega)^{\Theta=0} = 6/\sqrt{\pi}$ (naturally, not depending on $\omega$). Therefore, $a_{1,1}^{\Theta=0} = 0$, $a_{0,1}^{\Theta=0} = 6/\sqrt{\pi}$. This gives the desired relation between the pole parts, in agreement with the result of Exercise 11.2 obtained for different asymptotic conditions on the background fields.

## 11.5 Noncommutative Geometry and the Spectral Action Principle

A rigorous mathematical approach to noncommutative geometry was suggested by A. Connes. The idea is to formulate a set of axioms which any geometry should satisfy without relying on coordinate charts, and then abandon the requirement of commutativity. The approach allows one to give definitions of geometric structures when the very notion of a point is not well-defined, as in the case of Moyal spaces. Below we give a brief description of these ideas.

The central notion of noncommutative geometry is the *spectral triple* $(A, \mathscr{H}, \not{D})$ consisting of an algebra $A$, a Hilbert space $\mathscr{H}$, and a Dirac operator $\not{D}$. Let us consider the elements of the spectral triple one by one.

We start with the algebra. Clearly, continuous functions on a manifold $\mathscr{M}$ form a commutative associative algebra with respect to the point-wise product. Therefore, to any $\mathscr{M}$ one can associate an algebra. According to the Gelfand-Naimark theorem converse is also true: any associative commutative algebra (with some restrictions, as e.g., being a $C^*$ algebra, which are not important for further discussion) is an algebra of continuous functions on some manifold. An algebra defines a manifold in this sense. Let us now abandon the restriction that the algebra is commutative. One can then say that a noncommutative associative algebra $A$ (with some additional requirements) defines a noncommutative manifold. In the discussion below we shall freely switch between continuous and smooth functions, which is a relatively harmless procedure.

As we know, dynamical fields over a manifold belong to a vector bundle over this manifold, and square integrable sections (fields) form a Hilbert space. Therefore, an abstract Hilbert space $\mathscr{H}$ can be used to describe dynamical variables. Internal background geometry of a manifold is described by background fields, which may be a metric or a gauge field, for example. Such fields enter naturally the Dirac operator which acts on $\mathscr{H}$. If $\mathscr{H}$ is the space of square integrable spinors, and $\not{D}$ is the standard Dirac operator, we return to the situation which appeared in the previous Chapters.

To be able to say that abstract objects $(\mathscr{A}, \mathscr{H}, \not{D})$ indeed represent generalizations of corresponding notions in the commutative geometry to noncommutative setting, one has to impose certain restrictions, which are called the Axioms of Spectral Triples. The axioms are the following:

(a) the algebra $A$ is represented on $\mathscr{H}$ by bounded operators,
(b) $\not{D}$ is an unbounded self-adjoint operator on $\mathscr{H}$ such that for every $a \in A$ the operators $a(\not{D} \pm i)^{-1}$ are compact,
(c) for every $a \in A$ the operators $[\not{D}, a]$ are bounded.

To discuss the meaning of these axioms one has to use definitions of bounded and compact operators, see Sect. 3.3.

If $A = C^\infty(\mathbb{R}^n)$, and $\mathscr{H}$ is the space of square integrable spinors on $\mathbb{R}^n$, the point-wise multiplication by functions from $A$ maps $\mathscr{H}$ to itself, and this is obviously a bounded operator. The axiom (a) requests the same property from the star product.

The partial derivative $i\partial_\mu$ multiplies each $\exp(ikx)$ by $-k_\mu$ and is therefore unbounded. The same holds for the standard Dirac operator $i\gamma^\mu\partial_\mu$ on the space of square integrable spinors, which is also unbounded, as in the first part of the axiom (b). The operator $a\not{D}^{-1}$ provides a "regularization at high momenta" (which is necessary to be compact) in the usual commutative case, modulo possible problems with the invertibility. To avoid these problems, one adds an imaginary number $\pm i$, which definitely does not belong to the spectrum of a selfadjoint operator. The axiom (b) requests that this construction works in the noncommutative case as well. However, an even better regularization at high momenta can be achieved if one uses a higher order elliptic differential operator instead of the Dirac operator. This is excluded by the axiom (c). Indeed, the commutator with a multiplication operator can "eat" one derivative, so that $[a, i\gamma^\mu\partial_\mu] = -i\gamma^\mu(\partial_\mu a)$ becomes a bounded operator, but it cannot eat two derivatives. The axiom (c) is a requirement (in an abstract language) that $\not{D}$ is first order.

It should be emphasized that the comments above are intended to give an idea of what a spectral triple looks like. In any particular case a proof that the axioms (a)–(c) are satisfied may be rather tedious. For example, a rigorous proof that the Moyal plane is a spectral triple can be found in [126], and it is far from being trivial.

Suppose $\not{D}$ depends on some NC background fields $\phi$. Given a spectral triple, one can define an action for $\phi$ by using the spectral action principle of Chamseddine and Connes. The idea is to define a functional of background fields $\phi$ in terms of a spectral function of $\not{D}(\phi)$. The definition is the following:

$$I_{\text{spec}}[\phi] = \text{Tr}\big[\chi(\not{D}(\phi)/\Lambda)\big]. \tag{11.46}$$

Spectral action (11.46) depends on a cutoff function $\chi$ (which is usually supposed to be symmetric) and a cutoff parameter $\Lambda$. In the limit $\Lambda \to \infty$, the spectral action can be decomposed through the heat kernel of $\not{D}^2$,

$$I_{\text{spec}}[\phi] \simeq \sum_k \Lambda^{n-2k}\chi_{2k}a_{2k}(\not{D}^2), \tag{11.47}$$

where the coefficients $\chi_{2k}$ are defined through the Laplace transform of $\chi$.

If $\not{D}$ is just the usual Dirac operator on curved base manifold expansion (11.47) gives very reasonable results. The first term is the cosmological constant, the second one gives the Einstein action and etc. Interestingly, even the boundary terms in the gravity action come out correctly [60]. In this regard the spectral action principle is quite similar to the idea by Zeldovich and Sakharov that gravity action may be entirely induced by quantum effects of fields propagating on a given space-time. That is, the classical action of background fields is an effective action of quantum excitations over the given background. In this case the cosmological constant, the Newton constant are induced constants which are determined in terms of parameters of dynamical fields and the cutoff $\Lambda$. In the same way one can induce the action of gauge and other background fields.

As has been explained in Sect. 11.3 the Moyal multiplication by a smooth function *is* a bounded operator. Let us complete the spectral triple by a Dirac operator having some interesting physical content. The electromagnetic field on the Moyal space can be described by the Dirac operator of the form [127]

$$\not{D} = -i\gamma^\mu(\partial_\mu + iL(A_\mu) - iR(A_\mu)), \tag{11.48}$$

which, in addition to the axioms formulated above, satisfies also some reality conditions. According to (11.47), to evaluate the spectral action in leading order it is enough to calculate the first non-vanishing heat kernel coefficient. Let us do this on a four-dimensional Euclidean Moyal plane. The squared Dirac operator reads

$$\begin{aligned}\not{D}^2 = &-g^{\mu\nu}(\partial_\mu + iL(A_\mu) - iR(A_\mu))(\partial_\nu + iL(A_\nu) - iR(A_\nu))\\ &-\frac{i}{4}(L(F_{\mu\nu}) - R(F_{\mu\nu}))[\gamma^\mu, \gamma^\nu],\end{aligned} \tag{11.49}$$

where we introduced a constant 'metric' $g^{\mu\nu}$ for a technical purpose. The method we are going to use is algebraic and based on properties of the traces like (11.15). It is easy to show that

$$\begin{aligned}&\mathrm{Tr}[(L(l)R(r))^{\mu_1\dots\mu_m}\partial_{\mu_1}\dots\partial_{\mu_m}e^{-t\Delta}]\\ &\quad = i^m G^{(m)}_{\mu_1\dots\mu_m}\mathrm{Tr}[(L(l)R(r))^{\mu_1\dots\mu_m}e^{-t\Delta}],\end{aligned} \tag{11.50}$$

where, for odd $m$ the coefficients $G^{(m)}$ vanish, while for even $m$ they may be obtained by consecutive differentiation of (11.15) with respect to $g^{\mu\nu}$. One has to keep in mind that (11.15) is modified by the volume element $\sqrt{g}$ under the integral, and that $g^{\mu\nu}$ is symmetric, so that not all of the components are indeed independent. For example,

$$\frac{\delta}{\delta g^{\mu\nu}} g_{\rho\sigma} = -\frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}).$$

In particular, one has

$$\begin{aligned}G^{(2)}_{\mu\nu} &= \frac{\sqrt{g}}{2t}g_{\mu\nu},\\ G^{(4)}_{\mu\nu\rho\sigma} &= \frac{\sqrt{g}}{4t^2}(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}).\end{aligned} \tag{11.51}$$

To evaluate the heat trace one has to represent $-t\not{D}^2 = A + B$, where $A = -t\Delta$, and $B$ is the rest. Next, one expands $e^{-t\not{D}^2}$ in $B$. Note, that $A$ and $B$ do not commute and the Duhamel formula (5.61) should be used, which gives

$$\begin{aligned}e^{A+B} = e^A\Big(&1 + B + \frac{1}{2}[B,A] + \frac{1}{2}B^2 + \frac{1}{6}\big[[B,A],A\big] + \frac{1}{3}[B,A]B\\ &+ \frac{1}{6}B[B,A] + \frac{1}{24}\big[\big[[B,A],A\big],A\big]\\ &+ \frac{1}{8}\big[[B,A],A\big]B + \frac{1}{8}[B,A]^2 + \mathscr{O}(B^3)\Big).\end{aligned} \tag{11.52}$$

Now we have all the instruments to make the calculations, which are rather straightforward, though lengthy. The result is that $a_0$ does not depend on $A_\mu$, $a_2 = 0$, and

$$a_4(\not{D}^2) = \frac{1}{12\pi^2} \int \sqrt{g}\, d^4x\, F_{\mu\nu}^2, \tag{11.53}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i A_\mu \star A_\nu - i A_\nu \star A_\mu \tag{11.54}$$

is the noncommutative field strength. Note, that in contrast to the commutative case there are quadratic terms.

Due to (11.47), the leading term in the spectral action is proportional to (11.53). Although in the commutative limit the field $A_\mu$ disappears from Dirac operator (11.48), the spectral action remains non-trivial in this limit and becomes just the Maxwell action! Note, that the heat kernel coefficient (11.53) in this limit is twice the corresponding coefficient for commutative case (7.96).

Noncommutative physics and noncommutative mathematics are very active areas of research with a constantly changing landscape. For this reason, we do not include any concluding section to this Chapter. Application of the spectral theory methods to quantum field theory is a continuing story.

## 11.6 Literature Remarks

Standard reviews on NC field theory are [86, 236]. These references contain an extensive literature survey on the history and applications of noncommutativity. The Doplicher-Fredenhagen-Roberts approach to noncommutative space-times was suggested in [84, 85], for review see [206]. One can also find here arguments why classical gravity effects prevent coordinates from being simultaneously measurable with arbitrary high precision.

The Moyal formula (11.2) is also called the Gronewold or the Weyl formula.

We have not aimed here at presenting a quantum field theory on noncommutative spaces. Instead our purpose was to introduce elements of the spectral theory which are related to such quantized models. Short remarks on the quantization of the NC version (11.9) of Euclidean $\varphi^4$ model are the following. This model is not renormalizable on the Moyal plane in four dimensions at all orders. To make this model all-loop renormalizable one has to modify the kinetic term [146, 150]. Crucial differences between planar and non-planar cases are related to the phenomena of mixing between UV and IR scales in NC field theories, for a more detailed discussion see [13, 64, 190].

The heat kernel expansion on Moyal spaces was studied in [125, 127, 244, 246], see also a mini-review [248]. If one has compact NC dimensions, or if $\theta$ is degenerate, the behavior of non-planar contributions may differ considerably from what was described in Sect. 11.3. In particular, non-planar contributions to the heat kernel expansion on an NC torus depend crucially on the number-theory nature of the NC parameter [127].

Our treatment in Sect. 11.4 of the NC kink is borrowed from [172].

Section 11.5 deals with an area which was initially a part of pure mathematics. Its many technical tools are very similar to those of quantum field theory. Moreover, there are very interesting applications of the ideas of noncommutative geometry to particle physics [71]. Our exposition here is not quite rigorous and complete. An interested reader may refer the classical monograph by Connes [69] or other useful sources, such as [71, 141, 159].

There is a large variety of formulations of the Axioms of Spectral Triples in the literature. One can impose some additional requirements to make the spectral triple real, or orientable, or irreducible, or else. A fairly complete overview of various formulations can be found in [36].

The spectral action principle was proposed by Chamseddine and Connes [59]. For a review of applications of the NC geometry to particle physics, see [58]. Physical motivations for the spectral action principle are very similar to the idea that gravity may be entirely induced by quantum effects. The induced gravity principle was suggested by Zeldovich [261] and Sakharov [221]. A somewhat random choice of reviews on the induced gravity and its applications includes [3, 113, 201]. For a derivation of the Standard Model from the spectral action principle on a space with a "finite noncommutativity" see [61, 70, 71].

## 11.7 Exercises

**Exercise 11.1** Prove that Moyal product (11.2) is associative and satisfies property (11.5).

**Exercise 11.2** Calculate the heat kernel for operator (11.11) on a two-dimensional Moyal plane up to the order $\phi^2$ and up to zeroth order in explicit derivatives, i.e., the derivatives which do not enter the star product. Analyze possible ultraviolet divergences in this model and their renormalization.

**Exercise 11.3** Show that a Dirac operator with the leading part $\not{D} = i\gamma^a L(e_a^\mu)\partial_\mu$ cannot be used to form a spectral triple in the sense of Axioms from Sect. 11.5.

# Part IV
# Problem Solving

# Chapter 12
# Solutions to Exercises

## 12.1 Chapter 1. Geometrical Background

**1.1** First, we note that $\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}(\delta g_{\rho\sigma})$. Then it is easy to prove by inspection that the variation of $\Gamma$ under arbitrary variation of the metric reads:

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}((\delta g_{\mu\sigma})_{;\nu} + (\delta g_{\nu\sigma})_{;\mu} - (\delta g_{\mu\nu})_{;\sigma}). \tag{12.1}$$

The diffeomorphism variation of $\Gamma^{\rho}_{\mu\nu}$ follows from (1.81).

**1.2** To prove that $\varepsilon^{\mu_1\cdots\mu_n}$ is a tensor one has to demonstrate that it transforms properly under the diffeomorphism transformations. Let suppose that $\varepsilon^{\mu_1\cdots\mu_n}$ is indeed a tensor and derive the corresponding transformation rule. By Eq. (1.1)

$$\varepsilon'^{\mu'_1\cdots\mu'_n} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu'_n}}{\partial x^{\mu_n}}\varepsilon^{\mu_1\cdots\mu_n}. \tag{12.2}$$

The right hand side of this equation is a totally antisymmetric rank $n$ tensor. Therefore, it must be proportional to $\varepsilon^{\mu'_1\cdots\mu'_n}$. This yields

$$\varepsilon'^{\mu'_1\cdots\mu'_n} = \varepsilon^{\mu'_1\cdots\mu'_n}\frac{1}{n!}\varepsilon_{\nu'_1\cdots\nu'_n}\frac{\partial x^{\nu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\nu'_n}}{\partial x^{\mu_n}}\varepsilon^{\mu_1\cdots\mu_n} = \varepsilon^{\mu'_1\cdots\mu'_n}\det\left(\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right). \tag{12.3}$$

This is precisely the transformation rule which follows from the definition $\varepsilon^{\mu_1\mu_2\ldots\mu_n} = g^{-1/2}\tilde{\varepsilon}^{\mu_1\mu_2\ldots\mu_n}$.

**1.3** The line element on the two-sphere reads $(ds)^2 = r^2(d\theta^2 + \sin^2\theta d\varphi^2)$, where $\varphi \in [0, 2\pi[$, $\theta \in [0, \pi]$. The only non-zero components of the Christoffel connection are

$$\Gamma^{\theta}_{\varphi\varphi} = -\cos\theta\sin\theta, \qquad \Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = \tan\theta. \tag{12.4}$$

Due to the symmetry properties of the Riemann tensor it is enough to calculate

$$R^{\varphi}{}_{\theta\varphi\theta} = 1 \tag{12.5}$$

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Theoretical and Mathematical Physics,
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in order to define all other components and to prove (1.103). The Ricci tensor is proportional to the metric, $R_{\mu\nu} = g_{\mu\nu}$, so that $S^2$ is an Einstein manifold. The scalar curvature is a constant, $R = 2$.

**1.4** This is just a direct computation using the definitions.

**1.5** The key relation is

$$dr^* = \left(1 - \frac{2M}{r}\right)^{-1} dr. \tag{12.6}$$

The rest follows automatically.

**1.6** A direct computation of the stress-energy tensor for model (1.75) yields

$$T_{\mu\nu} = F_{\mu\rho} F_{\nu}{}^{\rho} + M^2 A_\mu A_\nu - \frac{1}{4} g_{\mu\nu} \big( F_{\lambda\rho} F^{\lambda\rho} + 2M^2 A_\lambda A^\lambda \big). \tag{12.7}$$

**1.7** First we note that the left hand side of (1.52) is nothing else than the commutator of two covariant derivatives acting on a vector with a single flat index. We calculate

$$[\nabla_\mu, \nabla_\nu] v^a = e^{\rho a} [\nabla_\mu, \nabla_\nu] v_\rho = -e^{\rho a} v_\sigma R^\sigma{}_{\rho\mu\nu} = -v_b R^{ba}{}_{\mu\nu}, \tag{12.8}$$

where we used (1.9). Equation (1.52) follows immediately.

**1.8** Equation (1.108) follows from the definition of $\beta$, Clifford relation (1.61), and the properties of the Lorentzian gamma matrices $(\gamma_a)^\dagger = -\gamma_a$, if $a = 0$, and $(\gamma_a)^\dagger = \gamma_a$, if $a \neq 0$.

**1.9** To find the generators, let us rewrite (1.62) in an infinitesimal form with $S = 1 + s + \cdots$, $\Lambda^a{}_b = \delta^a_b + \lambda^a{}_b + \cdots$. We have the equation

$$[s, \gamma^a] = \lambda^a{}_b \gamma^b, \tag{12.9}$$

which is solved by

$$s = -\frac{1}{8} \lambda_{ab} [\gamma^a, \gamma^b], \tag{12.10}$$

thus giving us the desired form of the generator $s$.

**1.10** Let us restrict ourselves to the transformations $S$ which can be connected to the unit element of the group $Spin(1, n-1)$. They are exponents of the generators, i.e., $S = \exp(s)$. By using (1.108) and (12.10), one finds

$$s^\dagger = -\frac{1}{8} \lambda_{ab} [\gamma^{b\dagger}, \gamma^{a\dagger}] = \frac{1}{8} \lambda_{ab} [\beta \gamma^a \beta, \beta \gamma^b \beta] = -\beta s \beta.$$

Consequently,

$$S^{-1} = \exp(-s) = \exp(\beta s^\dagger \beta) = \beta S^\dagger \beta.$$

**1.11** Let us take the so-called Dirac representation of the gamma matrices

$$\gamma^0 = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^a = i\begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \tag{12.11}$$

where $a = 1, 2, 3$, and $\sigma^a$ are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12.12}$$

In this representation $\beta = i\gamma^{a=0}$ and $\gamma_{a=2}$ are real and symmetric matrices while $\gamma^1$ and $\gamma^2$ are antisymmetric. One can choose $C = \beta\gamma^{a=2}$. Obviously $C^* = C$ and $C^T = C^{-1} = -C$. One then easily proves (1.67). Another useful relation is $(C\beta)\gamma^a(C\beta) = \gamma^{a*}$. Consider an infinitesimal $Spin(1,3)$ transformation of $\psi$,

$$\delta\psi = s\psi.$$

Then

$$\delta\psi^c = C\overline{(s\psi)}^T = C\beta s^*\psi^* = sC\beta\psi^* = s\psi^c,$$

which means that the spinor $\psi^c$ belongs to the same representation as $\psi$. Consequently, the Majorana condition $\psi = \psi^c$ is Lorentz covariant.

**1.12** The interaction of a spinor field with a vector field is described by the term $A_\mu\bar{\psi}\gamma^\mu\psi$, see (1.73). The matrix $\gamma^0\gamma^\mu$ is always Hermitian. In the Majorana representation it is also real and, hence, symmetric. Besides, the Majorana spinors are real, $\psi^\dagger = \psi^T$. Therefore, the interaction term written above is a symmetric form computed on two identical anticommuting spinors. Such forms vanish identically.

On the same grounds, the $\gamma^0$ matrix in the Majorana representation is antisymmetric. Therefore, on *commuting* Majorana spinors the mass term vanishes.

**1.13** In this case the coordinate on the boundary is $\varphi$. The inward pointing unit vector has only one non-zero component $n_\theta = -1$. By using (12.4) one immediately obtains

$$K_{\varphi\varphi} = \sin\theta_0\cos\theta_0 = \frac{1}{\sin\theta_0}g_{\varphi\varphi}. \tag{12.13}$$

For $\theta_0 \to 0$ the last equation becomes the familiar expression for the extrinsic curvature of the 2-ball (1.92). For $\theta_0 = \pi/2$ the extrinsic curvature is indeed zero. The equatorial circle is a totally geodesic submanifold in $S^2$ (and indeed it is a geodesic line in $S^2$).

**1.14** One finds with the help of (1.90) $K^+_{ij} = K^-_{ij} = -\frac{\alpha}{2}g_{ij}$ where $g_{ij}$ is the metric on the brane.

**1.15** Consider variation

$$\delta\int_{\mathscr{M}} d^n x\,\sqrt{-g}(R-2\Lambda)$$
$$= \text{bulk t.} - \int_{\partial\mathscr{M}} d^{n-1}x\,\sqrt{h}\,n^\lambda\big(\nabla^\mu\delta g_{\lambda\mu} - \nabla_\lambda\big(g^{\mu\nu}\delta g_{\mu\nu}\big)\big), \qquad (12.14)$$

where 'bulk t.' denote the bulk terms, $n^\mu$ is the inward pointing unit vector to $\partial\mathscr{M}$. To get (12.14) we used (1.107). Note that one can always choose a special coordinate system (fix a gauge) where the following variations of the metric on $\partial\mathscr{M}$ vanish:

$$n^\lambda\delta g_{\lambda\mu} = 0, \qquad n^\lambda n^\mu\nabla_\mu\delta g_{\lambda\nu} = 0. \qquad (12.15)$$

Embedding of $\partial\mathscr{M}$ can be described by the equation $f(x) = 0$. A function $f$ determines the normal vector $n_\mu = \chi f_{,\mu}$, where $\chi$ is a normalization coefficient, $\chi^{-2} = g^{\mu\nu} f_{,\mu} f_{,\nu}$. It is implied that the embedding equation remains unchanged under variations of the metric. Then, it follows from (12.15) that in the chosen coordinates $\delta\chi = 0$. Therefore, $\delta n_\mu = 0$ as well.

By using this fact one can proceed with the r.h.s. of (12.14)

$$\delta\mathscr{S} \equiv -\int_{\partial\mathscr{M}} d^{n-1}x\,\sqrt{h}\,n^\lambda\big(\nabla^\mu\delta g_{\lambda\mu} - \nabla_\lambda\big(g^{\mu\nu}\delta g_{\mu\nu}\big)\big)$$
$$= -\int_{\partial\mathscr{M}} d^{n-1}x\,\sqrt{h}\,\big(-\delta g_{\lambda\mu}(\nabla^\mu n^\lambda) - g^{\mu\nu}\big(n^\lambda\nabla_\lambda\delta g_{\mu\nu}\big)\big)$$
$$= -\int_{\partial\mathscr{M}} d^{n-1}x\,\sqrt{h}\big(\delta h_{\lambda\mu}K^{\mu\lambda} - g^{\mu\nu}\big(n^\lambda\nabla_\lambda\delta g_{\mu\nu}\big)\big), \qquad (12.16)$$

where we used the definition of extrinsic curvature (1.87). Consider now variation of the trace of the extrinsic curvature:

$$\delta K = -\delta h^{\mu\nu} n_{\mu;\nu} - h^{\mu\nu}\delta n_{\mu;\nu} = -\delta h^{\mu\nu} n_{\mu;\nu} + h^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} n_\lambda$$
$$= -\frac{1}{2} h^{\mu\nu} n^\lambda \nabla_\lambda\delta g_{\mu\nu}. \qquad (12.17)$$

By taking into account (12.17) and the second condition in (12.15) one can rewrite (12.16) as

$$\delta\mathscr{S} = -\int_{\partial\mathscr{M}} d^{n-1}x\,\sqrt{h}\big(\delta h_{\lambda\mu}K^{\mu\lambda} + 2\delta K\big)$$
$$= -2\delta\left[\int_{\partial\mathscr{M}} d^{n-1}x\,\sqrt{h}K\right] - \int_{\partial\mathscr{M}} d^{n-1}x\,\sqrt{h}\,\delta h_{\lambda\mu}(K^{\lambda\mu} - h^{\lambda\mu}K). \qquad (12.18)$$

With the help of (12.18) one can find variation of (1.109)

$$\delta\tilde{I}_{EH}[g] = \text{bulk t.} - \frac{1}{16\pi G_N}\int_{\partial\mathscr{M}} d^{n-1}x\,\sqrt{h}\,\delta h_{\lambda\mu}(K^{\lambda\mu} - h^{\lambda\mu}K) \qquad (12.19)$$

If the metric on $\partial\mathscr{M}$ is fixed, $\delta h_{\mu\nu} = 0$, the variation of the modified action (1.109) vanishes provided that the metric in the bulk is a solution to the Einstein equations (1.21).

**1.16** The action in the presence of the brane can be written in following form:

$$I[\varphi, g] = \frac{1}{16\pi G_N}\int_{\mathscr{M}} d^n x\,\sqrt{-g}(R-2\Lambda) + \frac{1}{8\pi G_N}\int_{\partial\mathscr{M}} d^{n-1}x\,\sqrt{h}(K_+ + K_-) + I_{\text{brane}}[\varphi, h]. \quad (12.20)$$

The second term in the r.h.s. of (12.20) contains extrinsic curvatures on the both sides of the brane. This term is introduced to take into account the defect of the geometry and ensure the correct variational procedure. The variation of the action follows from formula (12.19)

$$\delta I[\varphi, g] = \text{bulk t.} - \frac{1}{16\pi G_N}\int_{\partial\mathscr{M}} d^{n-1}x\,\sqrt{h}\,\delta h_{\lambda\mu} \times \left[(K_+^{\lambda\mu} - h^{\lambda\mu}K_+) + (K_-^{\lambda\mu} - h^{\lambda\mu}K_-)\right] + \delta_h I_{\text{brane}}[\varphi, h]. \quad (12.21)$$

This immediately yields the junction conditions (1.110).

**1.17** The two-dimensional gamma-matrices $\gamma^\mu$ can be defined with the help of the Pauli matrices $\sigma_i$. One can choose such a representation of $\gamma^\mu$ that the spinor covariant derivative takes the form $\nabla_\mu\psi = \partial_\mu\psi + \frac{i}{2}\sigma_3\omega_\mu\psi$ where $\omega_\mu dx^\mu = -\sin\theta\; d\tau$. A pair of explicit solutions to (1.112) is

$$\epsilon_1(\tau,\theta) = e^{i\tau/2}\begin{bmatrix}\sin(\theta/2+\pi/4)\\ -\cos(\theta/2+\pi/4)\end{bmatrix}, \qquad \epsilon_2(\tau,\theta) = e^{-i\tau/2}\begin{bmatrix}\cos(\theta/2+\pi/4)\\ \sin(\theta/2+\pi/4)\end{bmatrix}, \quad (12.22)$$

where $\tau$ and $\theta$ are coordinates on $S^2$, see (1.97). The Killing spinors obey anti-periodic boundary conditions

$$\epsilon_i(\tau+2\pi,\theta) = -\epsilon_i(\tau,\theta), \quad (12.23)$$

and are normalized as $\epsilon_i^\dagger\epsilon_j = \delta_{ij}$ with $i, j = 1, 2$. There are 3 Killing vectors on $S^2$. One can check with the help of (1.112) that the following vectors:

$$(V^0)_\mu = \epsilon_1^\dagger\gamma_\mu\epsilon_1 = -\epsilon_2^\dagger\gamma_\mu\epsilon_2, \qquad (V^+)_\mu = ((V^-)_\mu)^* = \epsilon_2^\dagger\gamma_\mu\epsilon_1 \quad (12.24)$$

satisfy Killing equation (1.82). The vectors can be written as $V_\mu = \epsilon_{\mu\nu}\nabla^\mu\varphi$ where $\varphi$ are 3 dipole eigenmodes of the scalar Laplacian with the eigenvalues $l(l+1) = 2$. The dipole modes are $\varphi^0 = \sin\theta$, $\varphi^\pm = \cos\theta e^{\pm i\tau}$.

## 12.2 Chapter 2. Quantum Fields

**2.1** On a constant time hypersurface $t = const$ the future-directed normal vector is $n^\mu = -g^{\mu 0}/\sqrt{|g_{00}|}$ and so $d\Sigma^\mu = -g^{\mu 0}/\sqrt{g^{00}}(\det g_{ij})^{1/2}\, d^d x$. To prove the required formula one has to take into account that $\det g_{ij} = |g^{00}| \det g$, hence $d\Sigma^\mu = -g^{\mu 0}\sqrt{g}\, d^d x$. Then one gets

$$Q_1 - Q_2 = \int_{\Sigma_1} d\Sigma^\mu\, j_\mu - \int_{\Sigma_2} d\Sigma^\mu\, j_\mu = \int_B \sqrt{g}\, d^n x\, \nabla^\mu j_\mu = 0,$$

where $Q_1$ and $Q_2$ are the integrals over undeformed and deformed surfaces $\Sigma_1$ and $\Sigma_2$, respectively. $B$ is the volume bounded by $\Sigma_1 \cup \Sigma_2$.

**2.2** The conditions on the coefficients in (2.92), (2.93) are the following:

$$\sum_k \bar{\alpha}_{ik}^{(\pm)} \alpha_{jk}^{(\pm)} - \sum_p \bar{\beta}_{ip}^{(\pm)} \beta_{jp}^{(\pm)} = \pm\delta_{ij}, \tag{12.25}$$

$$\sum_k \bar{\alpha}_{ik}^{(+)} \alpha_{jk}^{(-)} - \sum_p \bar{\beta}_{ip}^{(+)} \beta_{jp}^{(-)} = 0, \tag{12.26}$$

where the bar denotes the complex conjugation. Conditions (12.25) guarantee (2.20), while (12.26) ensure (2.19).

The Bogoliubov transformation to a new set of creation and annihilation operators follows from (2.23), (2.92), (2.93),

$$\tilde{a}_i = \langle \tilde{f}_i^{(+)}, \phi \rangle = \sum_k \bar{\alpha}_{ik}^{(+)} a_k - \sum_p \bar{\beta}_{ip}^{(+)} b_p^+, \tag{12.27}$$

$$\tilde{b}_j^+ = -\langle f_j^{(-)}, \phi \rangle = \sum_p \bar{\beta}_{jp}^{(-)} b_p^+ - \sum_k \bar{\alpha}_{jk}^{(-)} a_k. \tag{12.28}$$

The constants $\alpha$'s and $\beta$'s in (12.27), (12.28) are called the Bogoliubov coefficients.

It is easy to check that vectors (2.28) in the Fock space are the eigenvectors of operators $N_i(a) = a_i^+ a_i$, $N_j(b) = b_j^+ b_j$. The corresponding eigenvalues are numbers of quantum excitations of different sorts in the given state. The number of new particles in the vacuum state $|0\rangle$, see (2.27), is

$$\langle 0|\tilde{a}_i^+ \tilde{a}_i|0\rangle = \sum_p |\beta_{ip}^{(+)}|^2, \qquad \langle 0|\tilde{b}_j^+ \tilde{b}_j|0\rangle = \sum_k |\alpha_{jk}^{(-)}|^2. \tag{12.29}$$

The vacuum does not change if the Bogoliubov transformations do not mix creation and annihilation operators of different sorts.

**2.3** The proof that two quantization approaches coincide is analogous to the proof given for the scalar theory in Minkowski space-time in Sect. 2.4. One has to take into account that the expression for $d\Sigma^\mu$ obtained in Exercise 2.1 and find by using (2.30) the canonical momentum

$$\pi(t,x) = -\sqrt{g}\big(D^0\varphi\big)^+(t,x).$$

The product (2.9) can be written as

$$\langle f_1, f_2 \rangle = i(f_1, \pi_2^*) - i(\pi_1^*, f_2), \tag{12.30}$$

where $\pi_k = -\sqrt{g}\, D^0 f_k$ and

$$(f, g) = \int_\Sigma d^d x\, f^* g \tag{12.31}$$

is a standard inner product. (It should be noted that (12.31) is defined with a non-invariant measure.) One also finds that

$$\varphi(f_k) = i(f_k, \pi^+) - i(\pi_k^*, \varphi). \tag{12.32}$$

Because $f_k$ and $\pi_k$ are independent Cauchy data on $\Sigma$ it follows from (2.18) that

$$\left[(\pi_1^*, \varphi), (f_2^*, \pi)\right] = i(\pi_1^*, f_2). \tag{12.33}$$

The latter is equivalent to the local commutator

$$\left[\varphi(x), \pi(y)\right] = i\delta^d(x-y) \tag{12.34}$$

in the canonical approach.

**2.4** To prove (2.94) for Bose fields one can consider a scalar model and use results of Exercise 2.3. Let $f_1$, $f_2$ be two solutions to the wave equations. We choose the Cauchy data on $\Sigma$ such that $f_k|_\Sigma = 0$ and $\partial_n f_k|_\Sigma \equiv \chi_k \neq 0$. We also assume that functions $\chi_1$, $\chi_2$ have compact supports on $\Sigma$ in a small neighborhood of points $x_1$ and $x_2$, respectively. It follows from (12.30) that $\langle f_1, f_2\rangle = 0$. The commutator (2.94) vanishes as a consequence of postulate (2.18).

**2.5** The equations of motion for model (2.95) are

$$\nabla^2 A_\mu - R_\mu^\nu A_\nu - M^2 A_\mu = 0. \tag{12.35}$$

It follows from Eqs. (1.76) for model (1.75) that there is constraint $\nabla A = 0$. If $\nabla A = 0$ Eqs. (1.76) reduce to (12.35). Therefore, solutions to (1.76) are a subclass of solutions to (12.35) determined by the condition $\nabla A = 0$. Because of this condition the vector field in (1.76) has $n-1$ independent components in $n$-dimensional space-time, while the vector field in (2.95) is unconstrained.

The unconstrained vector field can be uniquely decomposed onto transverse, $A^\perp$, and longitudinal, $A^\parallel$, components,

$$A = A^\perp + A^\parallel, \qquad A^\parallel = \nabla\varphi, \qquad \nabla A^\perp = 0, \tag{12.36}$$

where $\nabla^2\varphi = \nabla A$. It follows from (12.35) that

$$(\nabla^2 - M^2)\nabla A = 0, \tag{12.37}$$

and, hence, $\varphi = M^{-2}\nabla A$.

The difference between (1.75) and (2.95) is in longitudinal components. Let us consider the relativistic product on a space of solutions to (12.35). The product is

$$\langle A_1, A_2\rangle^{(1)} = \int_\Sigma d\Sigma^\mu\, j_\mu^{(1)}(A_1, A_2), \tag{12.38}$$

$$j^{(1)}_\mu(A_1, A_2) = i(A^\nu_1)^* \nabla_\mu A_{2\nu} - i A^\nu_2 \nabla_\mu (A_{1\nu})^*. \quad (12.39)$$

If $A_1$ and $A_2$ are solutions to (12.35) the current $j^{(1)}_\mu(A_1, A_2)$ is divergence free. By using (12.38) one can construct a quantum theory of unconstrained fields with commutation relations

$$[A(f_1), A(f_2)] = \langle f_1, f_2\rangle^{(1)}. \quad (12.40)$$

The relativistic product in model (1.75) is defined in terms of the current $j_\mu(A_1, A_2)$, see (2.14). A direct check shows that

$$j_\mu(A_1, A_2) = j^{(1)}_\mu(A_1, A_2) + i\nabla A^*_1 A_{2\mu} - i\nabla A_2 A^*_{2\mu} + \nabla^\nu Q_{\mu\nu}(A_1, A_2), \quad (12.41)$$

$$Q_{\mu\nu}(A_1, A_2) = i(A^*_{1\mu} A_{2\nu} - A^*_{1\nu} A_{2\mu}). \quad (12.42)$$

The last term in the r.h.s. of (12.41) yields a total divergence in the product $\langle A_1, A_2\rangle$. Thus, for transverse solutions which vanish at spatial infinity or obey suitable boundary conditions the two inner products are equivalent,

$$\langle A^\perp, A^\perp\rangle = \langle A^\perp, A^\perp\rangle^{(1)}. \quad (12.43)$$

One concludes that a quantum theory of transverse vector fields in model (2.95) is equivalent to a quantum theory of vector model (1.75).

To see unphysical properties of (2.95) let us calculate the product of two longitudinal modes $(A^\parallel_k)_\mu = \partial_\mu \varphi_k$ where $\varphi_k$ are two scalar functions which are solutions to (12.37). A straightforward calculation shows that

$$\begin{aligned}
&\langle A^\parallel_1, A^\parallel_2\rangle^{(1)} \\
&\quad = i\int_\Sigma d\Sigma^\mu \big[-M^2\big(\varphi^*_1 \nabla_\mu \varphi_2 - \nabla_\mu \varphi^*_1 \varphi_2\big) + \nabla^\nu \big(\nabla_\nu \varphi^*_1 \nabla_\mu \varphi_2 - \nabla_\mu \varphi^*_1 \nabla_\nu \varphi_2\big)\big] \\
&\quad = -M^2 \langle \varphi_1, \varphi_2\rangle,
\end{aligned} \quad (12.44)$$

where we used (12.37) and omitted a total divergence. Therefore, the relativistic product of longitudinal components coincides up to a constant with the relativistic product $\langle \varphi_1, \varphi_2\rangle$ of scalar fields, see (2.9), (2.10). The important point is that the two products differ by the sign. This means that longitudinal components have a wrong norm (implying non-standard commutation relations) and they are unphysical degrees of freedom.

**2.6** In the massless case vector model (1.75) is invariant under the gauge transformations $A'_\mu = A_\mu + \nabla_\mu \lambda$. One can use these transformations to fix the gauge, i.e. make the components $A_\mu$ to obey certain conditions. We choose the Lorentz (or Feynman) condition $\nabla A = 0$, as an example of a coordinate covariant gauge.

The field equations in this gauge become

$$\nabla^2 A_\mu - R^\nu_\mu A_\nu = 0 \quad (12.45)$$

and coincide with (12.35) at $M = 0$, see Exercise 2.5. The difference between massive and massless theories is that the Lorentz gauge does not eliminate all unphysical degrees of freedom. The gauge transformations $A'_\mu = A_\mu + \nabla_\mu \lambda$ with $\nabla^2\lambda = 0$ leave (12.45) invariant. As one can see, these transformations are excluded if $M \neq 0$.

The consequence of the additional symmetry is that the massless vector field has one degree of freedom less then the massive field. In the Maxwell theory in $n$-dimensional space-time there are $n-2$ independent components of the gauge field (in four dimensions they are two polarizations of the photon). The reduction of degrees of freedom follows from the fact that the transverse solutions to (12.45) of the form $A^\perp = \nabla\lambda$, where $\nabla^2\lambda = 0$, have the vanishing norm

$$\langle A_1^\perp A_2^\perp \rangle^{(1)} = \langle A_1^\perp A_2^\perp \rangle = 0. \tag{12.46}$$

The inner product of $A^\perp = \nabla\lambda$ with any other transverse solution vanishes as well. Thus, such fields make no contribution to physical observables.

The remaining gauge freedom can be eliminated by fixing one of the components, say, by putting $A_0 = 0$ in some coordinate frame. In Minkowski space-time combination of the two conditions yields spatially transverse field $A_0 = 0$, $\partial^i A_i = 0$.

**2.7** If $n^\mu$ is a unit normal vector to a space-like hypersurface $\Sigma$ then the norm defined on $\Sigma$ is $\langle \psi, \psi \rangle = -i \int_\Sigma \sqrt{h}\, d^3x\, \bar{\psi} n^\mu \gamma_\mu \psi$, where $h$ is the determinant of the metric induced on $\Sigma$. Let us choose veilbeins $e_a^\mu$ such that $e_{a=0}^\mu = n^\mu$ on $\Sigma$. Then $n^\mu \gamma_\mu = \gamma_{a=0}$ and the product takes manifestly positive form $\langle \psi, \psi \rangle = \int_\Sigma \sqrt{h}\, d^3x\, \psi^+ \psi$.

The proof of (2.96) is straightforward.

**2.8** Representations (2.40), (2.41) can be found for each particular model with the help of the equations of motions on stationary backgrounds. The key point is that these equations are linear. It is convenient to express the relativistic products (2.9) in terms of fields and their canonical momenta.

**Scalar Fields** To get (2.40) for model (1.68) of a complex scalar field one can use (12.30), see Exercise 2.3. To derive (2.41) for real scalar field $\varphi$ one has to take into account the normalization of the action (compare with (1.68))

$$I[\varphi, g] = -\frac{1}{2} \int d^n x \sqrt{-g} \big(\partial_\mu \varphi \partial^\mu \varphi + m^2 \varphi^2\big). \tag{12.47}$$

The definition of the relativistic product for real scalars is determined by (2.9), (2.10).

**Spinor Fields** Consider model (1.73) on a stationary background. The canonical variables and conjugate momenta are $\psi$, $\pi = \frac{1}{2}\sqrt{g}\bar{\psi}\gamma_0$, $\bar{\psi}$, $\bar{\pi} = -\frac{1}{2}\sqrt{g}\gamma_0\psi$. The canonical Hamiltonian is

$$H[\psi] = \int d^{D-1}x (\pi \dot{\psi} + \dot{\bar{\psi}} \bar{\pi}) = \frac{i}{2} \int d^{D-1}x \sqrt{g} \bar{\psi} \gamma_0 \dot{\psi} + c.c., \tag{12.48}$$

and, if one takes into account (2.11),

$$H[\psi] = \frac{i}{2} \langle \psi, \dot{\psi} \rangle + c.c. \tag{12.49}$$

With the help of (1.67) formula (12.49) can be rewritten as

$$H[\psi] = \frac{i}{2}\big(\langle\psi,\dot{\psi}\rangle + \langle\psi^c,\dot{\psi}^c\rangle\big), \qquad (12.50)$$

which is equivalent to (2.40). Form (12.50) is invariant with respect to the charge conjugation operation.

**Vector Fields** The relativistic product for vector model (1.75) follows from (2.9) and (2.14),

$$\langle A_1, A_2\rangle = i(A_1,\pi_2) - i(\pi_1, A_2), \qquad (12.51)$$

$$(A,\pi) \equiv \int d^D x (A^\mu)^* \pi_\mu. \qquad (12.52)$$

Here $\pi^i = -\sqrt{g}F^{0i}$, $\pi^0 = 0$. In the canonical approach $\pi^i$ are the momenta conjugated to spatial components $A_i$ of the vector field. The canonical momentum $\pi^0$ conjugated to $A_0$ vanishes. With the help of (12.51) and equations of motion (1.76) one comes to (2.41).

**Non-Abelian Gauge Fields** The linearized equations of motion (1.79) can be rewritten in the equivalent form as

$$[D^\nu,[D_\nu,A_\mu]] - [D_\mu,[D^\nu,A_\nu]] - R_{\mu\nu}A^\nu + 2[A^\nu,F_{\nu\mu}] = 0. \qquad (12.53)$$

One can define the linearized action (which yields (12.53) after the variation) and find the canonical momenta $\pi^\mu$ conjugated to $A_\mu$. In the matrix form $\pi^\mu = -\sqrt{g}G^{0\mu}$ and $\pi^0 \equiv 0$. Then the canonical energy taken on solutions to (12.53) can be written as

$$H[A] = \frac{1}{2}(\pi,\dot{A}) - \frac{1}{2}(\dot{\pi},A), \qquad (12.54)$$

$$(f_1,f_2) \equiv -2\int dx^{D-1}\,\mathrm{Tr}\, f_1 f_2. \qquad (12.55)$$

If the background fields are stationary the energy can be represented as

$$H = \frac{i}{2}\langle A,\dot{A}\rangle, \qquad (12.56)$$

which agrees with (2.40). The product in (12.56) is defined by (2.9) and (2.15).

**2.9** Consider first a vector field with a non-zero mass $M$. Let $\omega_i$ and $\omega_i^{(1)}$ be the single-particle spectra for vector models (1.75) and (2.95), respectively. As was shown in Exercise 2.5, the set of $\omega_i$ is a subset in $\omega_i^{(1)}$ which corresponds to the transverse modes. To get $\omega_i$ from $\omega_i^{(1)}$ one has to exclude the energies $\omega_j^{(\|)}$ of the longitudinal components of the vector field. According to (12.37), $\omega_j^{(\|)}$ coincide with energies $\omega_i^{(0)}$ of a real scalar field in the given space-time. Thus, the spectral function of vector model (1.75) can be written in the form

$$\Phi = \sum_i f(\omega_i) = \Phi_{(1)} - \Phi_{(0)}. \tag{12.57}$$

Here $\Phi_{(1)}$ is the spectral function of model (2.95) and $\Phi_{(0)}$ is the same spectral function of a scalar field with equation $(\nabla^2 - M^2)\varphi = 0$.

The spectral function $\Phi$ for massless vector field (an Abelian gauge field) can be found in a similar way by using results of Exercise 2.6. In the Lorentz gauge one gets

$$\Phi = \Phi_{(1)} - 2\Phi_{(0)}. \tag{12.58}$$

Here $\Phi_{(1)}$ is the spectral function in the vector field theory described by (12.45) and $\Phi_{(0)}$ is the spectral function of a massless scalar field with the equation $\nabla^2\varphi = 0$. The difference between (12.58) and (12.57) is in additional subtraction of $\Phi_{(0)}$ which eliminates a contribution of transverse modes with zero norm, see (12.46). These modes, as we have seen, also obey the massless scalar field equation.

Representations (12.57), (12.58) turn out to be very convenient in applications. They enable one to reduce the problem of finding spectral functions of constrained fields to the same problem for unconstrained fields, which is easier.

**2.10** If the background field $B_\mu$ satisfies (1.78) Eqs. (12.53), see Exercise 2.8, are invariant under the following gauge transformations:

$$A'_\mu = A_\mu + [D_\mu, \lambda], \tag{12.59}$$

where $\lambda = \lambda(x)$ is an element of the corresponding Lie algebra. The gauge freedom (12.59) can be used to impose the Lorentz-like gauge

$$[D_\mu, A^\mu] = 0. \tag{12.60}$$

In this gauge (12.53) becomes

$$[D^\nu, [D_\nu, A_\mu]] - R_{\mu\nu}A^\nu + 2[A^\nu, F_{\nu\mu}] = 0. \tag{12.61}$$

The physical solutions are a subclass among solutions to (12.53) which obey (12.60). Such solutions can be called "transverse" and denoted as $A^\perp$. Like in case of the Abelian gauge theory there is a residual gauge symmetry among the transverse modes, $A_\mu \to A_\mu + [D_\mu, \lambda]$ provided that $[D, [D, \lambda]] = 0$.

The relativistic product on a space of solutions to (12.61) is

$$\langle A_1, A_2\rangle^{(1)} = \int_\Sigma j_\mu^{(1)}(A_1, A_2), \tag{12.62}$$

$$j_\mu(A_1, A_2) = -2i\,\mathrm{Tr}\big(A_1^\nu[D_\mu, A_{2\nu}] - [D_\mu, A_{1\nu}]A_2^\nu\big). \tag{12.63}$$

One can show that for the transverse solutions the two products, (12.62) and (2.15), coincide,

$$\langle A_1^\perp, A_2^\perp\rangle^{(1)} = \langle A_1^\perp, A_2^\perp\rangle. \tag{12.64}$$

Let us now describe the single-particle spectrum $\omega_i$ of solutions to (12.53) in the Lorentz-like gauge. Let $\omega_i^{(1)}$ be the single-particle energies corresponding to unconstrained solutions to (12.61). The physical spectrum is a subset in $\omega_i^{(1)}$. Consider a scalar multiplet $\varphi$ in the same representation of the Lie algebra which obeys the following equation on the background $B_\mu$

$$[D_\mu, [D^\mu, \varphi]] = 0. \tag{12.65}$$

One can prove that quantity $\varphi = [D, A]$ obeys (12.65) if $A_\mu$ is a solution to (12.61). Therefore, the spectrum $\omega_i^{(0)}$ of the single-particle frequencies of $\varphi$ yields the spectrum of solutions to (12.53) which are not transverse and should be eliminated.

Now one has to take into account that some transverse modes may have a vanishing norm. Such modes are $A_\mu^\perp = [D_\mu, \varphi]$ and $\varphi$ again obeys (12.65). It is not difficult to check that

$$\langle A_1^\perp, A_2^\perp \rangle^{(1)} = 0. \tag{12.66}$$

In this gauge there is a full analogy between non-Abelian and Abelian models. One concludes that the spectral function of the linearized non-Abelian perturbations is

$$\Phi = \Phi_{(1)} - 2\Phi_{(0)}, \tag{12.67}$$

where $\Phi_{(1)}$ is the spectral function in theory (12.61) and $\Phi_{(0)}$ is the spectral function of the massless scalar field with Eq. (12.65).

**2.11** The equation of motion for non-minimally coupled field is

$$\nabla^2 \varphi - (m^2 + \xi R)\varphi = 0. \tag{12.68}$$

One can use (1.107) to get the stress-energy tensor for model (2.98)

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu}\left(\phi_{,\rho}\phi^{,\rho} + m^2\phi^2\right) + \xi\left[\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)\phi^2 + g_{\mu\nu}(\phi^2)^{;\rho}_{;\rho} - (\phi^2)_{;\mu\nu}\right]. \tag{12.69}$$

On a static space-time the Killing field is orthogonal to constant time hypersurfaces $\Sigma$ and energy (2.38) takes the form

$$E = -\int_\Sigma T_0^0 \sqrt{-g}\, d^{n-1}x. \tag{12.70}$$

Hamiltonian (2.39) can be expressed as

$$H = \int_{\mathscr{B}} \mathscr{H} \sqrt{-g}\, d^{n-1}x, \tag{12.71}$$

where the Hamiltonian density is

$$\mathscr{H} = \frac{1}{2}\left(-g^{00}\phi_{,0}^2 + g^{ij}\phi_{,i}\phi_{,j} + (m^2 + \xi R)\phi^2\right). \tag{12.72}$$

By taking into account (12.69) one arrives at the following relation:

$$-T_0^0 = \mathscr{H} - \xi\big(R_0^0\phi^2 + g^{ij}(\phi^2)_{;ij}\big). \tag{12.73}$$

The last term in r.h.s. of (12.73) can be rewritten,

$$(\phi^2)_{;ij} = \Box\phi^2 - g^{00}(\phi^2)_{;00} = g^{00}\big((\phi^2)_{,0,0} - (\phi^2)_{;00}\big) + \frac{1}{\sqrt{-g}}\partial_i\big(\sqrt{-g}g^{ij}\partial_j\phi^2\big)$$
$$= \frac{1}{\sqrt{-g}}\partial_i\big(\sqrt{-g}g^{ij}\big((\phi^2)_{,j} - \phi^2 w_j\big)\big) + \nabla_\mu w^\mu \phi^2. \tag{12.74}$$

Here $w^\mu = \frac{1}{2}\nabla^\mu \ln|g_{00}|$ is a time-independent acceleration of the Killing observer. It can be shown that $\nabla_\mu w^\mu = -R_0^0$, so that for static space-times relation (12.73) takes the form

$$-T_0^0 = \mathscr{H} - \xi\frac{1}{\sqrt{-g}}\partial_i\big(\sqrt{-g}g^{ij}\big((\phi^2)_{,j} - \phi^2 w_j\big)\big) \tag{12.75}$$

which shows that $E$ and $H$ differ by a surface term.

**2.12** To prove (2.64) one has to take into account the simple identity

$$\partial_t[\theta(t-t')G^+(x,x') + \theta(t'-t)G^-(x,x')]$$
$$= \theta(t-t')\partial_t G^+(x,x') + \theta(t'-t)\partial_t G^-(x,x') \tag{12.76}$$

which follows from the fact that the commutator of field operators at coinciding time arguments vanishes. The second time derivative of (12.76) contains a term with the commutator of $\varphi(x)$ and $\partial\varphi(x)$. This term is responsible for the r.h.s. of (2.64).

**2.13** The single-particle modes for the massless field on a circle are

$$f_k^{(+)}(t,x) = \frac{1}{\sqrt{4\pi|k|}}e^{-ia(|k|t-kx)},$$

where $a = 2\pi/l$ and $k = \pm1, \pm2, \ldots$. The Wightman function for the field on a circle can be obtained by using definition (2.56)

$$G^+(t,x) = G^+(0,x^\mu) = \frac{1}{4\pi}\sum_{k=1}^{\infty}\frac{1}{k}(e^{ikau} + e^{ikav}) = \frac{1}{4\pi}(g(au) + g(av)), \tag{12.77}$$

where $u = t - x$, $v = t + x$ and

$$g(z) = \sum_{k=1}^{\infty}\frac{1}{k}e^{ikz}. \tag{12.78}$$

To take the sum we assume that $\Im t > 0$, $\Im z > 0$. Note that

$$f(z) = \sum_{k=1}^{\infty}e^{ikz} = -\frac{1}{2i}\left(\cot\frac{z}{2} + i\right), \tag{12.79}$$

$$g(z) = i \int^z dz' \, f(z') = -\ln\left(-2i \sin \frac{z}{2}\right) - \frac{i}{2} z. \tag{12.80}$$

The constant of integration in (12.80) is fixed by condition $g(i\tau) = 0$ at $\tau \to \infty$. By using (12.80) in (12.77) one gets (2.99).

**2.14** The Wightman function for the field on an interval is derived analogously to the Wightman function on a circle. By using definition (2.56) and single-particle modes

$$f_k^{(+)}(t, x) = \frac{1}{\sqrt{\pi k}} e^{-iakt} \sin akx,$$

where $a = \pi/l$ and $k = 1, 2, \ldots$, one finds

$$G^+(x, x') = -\frac{1}{4\pi}\Big(g\big(a(\Delta t + \mathbf{x} + \mathbf{x}')\big) + g\big(a(\Delta t - \mathbf{x} - \mathbf{x}')\big) \\ - g\big(a(\Delta t + \mathbf{x} - \mathbf{x}')\big) - g\big(a(\Delta t - \mathbf{x} + \mathbf{x}')\big)\Big), \tag{12.81}$$

where $\Delta t = t' - t$, $\mathrm{Im}\, t' > 0$. After taking into account (12.80), this yields

$$G^+(x, x') = \frac{1}{4\pi} \ln\left[\frac{\sin \frac{a}{2}(\Delta t + \mathbf{x} + \mathbf{x}') \sin \frac{a}{2}(\Delta t - \mathbf{x} - \mathbf{x}')}{\sin \frac{a}{2}(\Delta t + \mathbf{x} - \mathbf{x}') \sin \frac{a}{2}(\Delta t - \mathbf{x} + \mathbf{x}')}\right], \tag{12.82}$$

which coincides with (2.100). Equation (12.82) has a suitable form to check that $G^+$ is a solution to the wave equation on an interval.

**2.15** The Wightman functions (2.99), (2.100) can be used to derive canonical commutation relation. Indeed, as follows from definitions,

$$[\phi(t, \mathbf{x}), \dot{\phi}(t, \mathbf{x}')] = \partial_t' [G^+(x, x') - G^+(x', x)]_{t'=t}. \tag{12.83}$$

Consider field on a circle. From (2.99) we find

$$\partial_t' G^+(x, x') = -\frac{a}{8\pi}\left(\cot \frac{a}{2} u + \cot \frac{a}{2} v + 2i\right). \tag{12.84}$$

The identity (4.128) can be used to write (12.84) as

$$\partial_t' G^+(x, x') = -\frac{1}{4\pi}\left[\sum_{k=-\infty}^{\infty}\left(\frac{1}{u - lk} + \frac{1}{v - lk}\right) v + ia\right]. \tag{12.85}$$

The next step is to note that $\Im\, u = \Im\, \epsilon$ and

$$\frac{1}{x + i\epsilon} = \mathscr{P}\frac{1}{x} - i\pi\delta(x), \tag{12.86}$$

where the symbol $\mathscr{P}$ means the principal value. Thus, for r.h.s. of (12.83) we get

$$\partial_t' \big[G^+(x, x') - G^+(x', x)\big]_{t'=t} = i\delta_l(\mathbf{x} - \mathbf{x}'), \tag{12.87}$$

where $\delta_l(x)$ is a delta-function on a circle with of the length $l$

$$\delta_l(x) = \sum_{k=-\infty}^{\infty} \delta(x + kl). \tag{12.88}$$

It is easy to see that $\delta_l(x + l) = \delta_l(x)$.

The problem on an interval can be solved in the same way. One should note that the delta-function on the interval with the Dirichlet conditions can be written as

$$\delta_{l,D}(x,x') = \delta_{2l}(x-x') - \delta_{2l}(x+x'), \tag{12.89}$$

where $\delta_{2l}(x)$ is the delta-function on the circle of the length $2l$, see (12.88). It can be checked that $\delta_{l,D}(x,x')$ satisfies the Dirichlet conditions for the each arguments.

**2.16** With the help of (2.35), (2.65) one gets the following expression for the Wightman function of a massless scalar field:

$$G^+(x) = \frac{1}{(2\pi)^3}\int \frac{d\mathbf{p}}{2|\mathbf{p}|} e^{i(|\mathbf{p}|t - \mathbf{p}\mathbf{x})} = \frac{1}{4\pi^2 x}\int_0^\infty dp\, e^{ipt}\sin px = \frac{1}{4\pi^2 s_+^2}.$$

The integration implies that $\Im t = \epsilon > 0$. Therefore, the interval has to be defined as $s_+^2 \equiv -t^2 + x^2 - i\varepsilon(t)\epsilon$. Equation (2.68) follows if one applies rule (12.86). Expression for $G^-(x)$ is obtained by complex conjugation.

**2.17** First let us note that the Wightman function in the model in question is

$$G^+(0,x) = \frac{1}{(2\pi)^3}\int d\mathbf{p}\frac{1}{2\omega_\mathbf{p}} e^{i(\omega_\mathbf{p} t - \mathbf{p}\mathbf{x})}, \tag{12.90}$$

where prescription $\Im t > 0$ is assumed. Consider now integral (2.101)

$$\mathscr{G}(0,x) = \frac{1}{(2\pi)^4}\int d^4p\, \frac{e^{-ipx}}{p^2+m^2} \tag{12.91}$$

and the identity

$$\frac{1}{p^2+m^2} = -\frac{1}{2\omega_\mathbf{p}}\left(\frac{1}{p_0-\omega_\mathbf{p}} - \frac{1}{p_0+\omega_\mathbf{p}}\right). \tag{12.92}$$

Let us show how to use (12.91) to specify the Feynman propagator, see (2.60). We replace the denominator in (12.91) to $p^2+m^2+i\epsilon$, where $\epsilon > 0$ is a small parameter. At $t > 0$ integral (12.91) can be taken by making a closed contour in the upper complex plane of $p_0$, while at $t < 0$ the contour should be taken in the lower plane. The result of the integration will be determined by the poles of (12.92). Because of $i\epsilon$ term there will be a single pole in the upper plane at $p_0 = \omega_\mathbf{p} + i\epsilon$ and a single pole in the lower plane at $p_0 = -\omega_\mathbf{p} - i\epsilon$. This follows from the fact that at small $\epsilon$ expression $p^2 + i\epsilon$ is equivalent to $-p_0^2 + (\omega_\mathbf{p} + i\epsilon)^2$. By taking this into account we find from (12.91), (12.92)

$$\begin{aligned}\mathscr{G}(0,x) &= -\frac{i}{(2\pi)^3}\int d\mathbf{p}\frac{1}{2\omega_\mathbf{p}}\left[\theta(t)e^{i(\omega_\mathbf{p} t - \mathbf{p}\mathbf{x})} + \theta(-t)e^{-i(\omega_\mathbf{p} t + \mathbf{p}\mathbf{x})}\right] \\ &= -i(\theta(t)G^+(0,x) + \theta(-t)G^-(0,x)) = G^F(0,x).\end{aligned} \tag{12.93}$$

Analogously one can define advanced and retarded Green's functions. For instance, $G_A(0,x)$ requires to change $p_0$ in (12.91) to $p_0 - i\epsilon$, $\epsilon > 0$. Then at $t < 0$ the integral over $p_0$ can be replaced by a closed contour in the lower complex plane. In this region the denominator does not have poles and $G_A(0,x) = 0$.

Finally, let us dwell on the momentum representation of Green's functions which obey homogeneous equations. All of them are determined in terms of the Wightman function for which we can write

$$G^+(0,x) = \frac{1}{(2\pi)^3}\int d^4p\,\delta\big(p^2+m^2\big)\theta(p_0)e^{-ipx}. \tag{12.94}$$

It is easy to see that (12.94) coincides with (12.90).

**2.18** The Lorentz invariance of the Feynman propagator is obvious after taking into consideration results of Exercise 2.17. Consider the Lorentz invariance of other functions. To this aim it is enough to discuss the Wightman function. Consider (12.94). Except for the function $\theta(p_0)$ it has a manifestly covariant form. Thus we only need to show that $p_0$ does not change sign under the Lorentz transformations. To be more precise we consider transformations $y_\mu = \Lambda_\mu{}^\nu x_\nu$ that preserve the norm, $x^2 = y^2$, and among these transformation we are interested in a subgroup of $O(1,3)$ such that $\det\Lambda = +1$, $\Lambda_0{}^0 > 0$ (which consists of the so-called proper ortho-chronal Lorentz transformations). It is easy to understand that $\Lambda_0{}^0 = +\sqrt{1+\Lambda_0{}^i\Lambda_0{}^i}$. Because of the delta function in (12.94) we can assume that $p_0 = +\sqrt{\mathbf{p}^2+m^2}$. Therefore,

$$p_0' = \Lambda_0{}^\nu p_\nu = \Lambda_0{}^0 p_0 + \Lambda_0{}^i p_i \geq \Lambda_0{}^0 p_0 - (\Lambda_0{}^i\Lambda_0{}^i)^{1/2}(\mathbf{p}^2)^{1/2} > 0,$$

i.e., after the transformation the sign is left positive, if $p_0 > 0$.

**2.19** To prove that (2.104) is a solution to (2.102), (2.103) one has to use the fact that the Pauli-Jordan function is a solution to (2.63) with the initial conditions

$$G(x,y)|_{t_x=t_y} = 0, \qquad \partial_{t_x} G(x,y)|_{t_x=t_y} = -\delta(\mathbf{x}-\mathbf{y}),$$

which follow from definition (2.58) and commutation relations at coinciding times.

**2.20** The expression for the scalar gauge current follows from (1.72)

$$\langle J^\mu(x)\rangle = ie\lim_{x\to x'}\big((D_x^\mu)^* - D_{x'}^\mu\big)G^+(x,x'),$$

$$G^+(x,x') = \langle 0|\varphi^+(x)\varphi(x')|0\rangle.$$

The classical current for spinor field (1.73) is

$$J^\mu(x) = ie\bar\psi\gamma^\mu\psi.$$

The average of the quantum current is

$$\langle J^\mu(x)\rangle = ie\lim_{x\to x'}(\gamma^\mu)_a{}^b(G^+)^a{}_b(x,x'),$$

where

$$(G^+)^a{}_b(x,x') = \langle 0|\bar\psi^a(x)\psi_b(x')|0\rangle$$

is the Wightman function for the spinor field.

**2.21** The proof that (2.105) yields a solution to (2.84) is straightforward.

## 12.3 Chapter 3. Operators and Their Spectra

**3.1** For arbitrary $f_1$ and $f_2$ satisfying (3.37) we calculate

$$(Lf_1, f_2) - (f_1, Lf_2) = \int_0^{2\pi} dx^2 \big(-f_1^*(x^1 = 1) \cdot \partial_1 f_2(x^1 = 1) + \partial_1 f_1^*(x^1 = 1) \cdot f_2(x^1 = 1)\big), \tag{12.95}$$

where we used the fact that the surface integral at $x^1 = 0$ vanishes due to (fist) Dirichlet condition (3.37). By using second condition (3.37) we transform the right hand side of (12.95) as

$$\int_0^{2\pi} dx^2\, i\alpha \big(f_1^*(x^1 = 1) \cdot \partial_2 f_2(x^1 = 1) + \partial_2 f_1^*(x^1 = 1) \cdot f_2(x^1 = 1)\big). \tag{12.96}$$

There is no boundary in $x^2$ direction. One can integrate by parts in (12.96) to demonstrate that the two terms in the brackets cancel against each other. This proves that $L$ is symmetric.

To check that two sets (3.38) and (3.39) are orthogonal one has to show this for modes $f_{k,m}$ and $\bar{f}_{\bar{k},m}$, where $m = 0, \pm 1, \pm 2, \dots$. The scalar product of the two modes is

$$(f_{k,m}, \bar{f}_{\bar{k},m}) = 2\pi \int_0^1 dx^1 \cos kx_1 \cosh \bar{k}x_1 = \frac{\cos k \cosh \bar{k}}{k^2 + \bar{k}^2}(k \tan k + \bar{k} \tanh \bar{k}),$$

and it vanishes if one uses (3.40). Of course, orthogonality of the modes follows from general arguments based on the symmetry of the operator $L$.

**3.2** Obviously,

$$(\psi_1, \not{D}\psi_2) - (\not{D}\psi_1, \psi_2) = -i \int_{\partial\mathscr{M}} d^{n-1}x \sqrt{\tilde{g}}\, \psi_1^\dagger \gamma^n \psi_2. \tag{12.97}$$

The integrand in (12.97) vanishes on the boundary. Indeed,

$$\begin{aligned}\psi_1^\dagger \gamma^n \psi_2|_{\partial\mathscr{M}} &= \psi_1^\dagger \gamma^n (\Pi_D + (1 - \Pi_D))\psi_2|_{\partial\mathscr{M}} \\ &= \psi_1^\dagger \gamma^n (1 - \Pi_D)\psi_2|_{\partial\mathscr{M}} = \psi_1^\dagger \Pi_D \gamma^n \psi_2|_{\partial\mathscr{M}} = 0, \end{aligned}\tag{12.98}$$

where we used the property $\gamma^n(1 - \Pi_D) = \Pi_D \gamma^n$ (which follows from (3.49)), and $\psi_1^\dagger \Pi_D|_{\partial\mathscr{M}} = 0$ (which follows from boundary condition (3.47) on $\psi_1$).

**3.3** By using the commutation relations between covariant derivatives one can demonstrate that

$$\Delta^{(1)}(\varrho_{,\mu}) = (\Delta^{(0)}\varrho)_{,\mu}, \qquad \Delta^{(1)}(\epsilon_{\mu\nu}\varphi^{;\nu}) = \epsilon_{\mu\nu}(\Delta^{(0)}\varphi)^{;\nu}, \tag{12.99}$$

where $\Delta^{(0)}$ is scalar Laplacian (3.19). Obviously, the zero-modes of the scalar Laplacian corresponding to constant $\varrho$ and $\varphi$ do not contribute to the Hodge-de Rham decomposition. Thus, the transverse and longitudinal eigenvectors of the operator $\Delta^{(1)}$ are $\epsilon_{\mu\nu}\varphi_l^{;\nu}$, $\varphi_{l,\mu}$ where $l = 1, 2, \dots$ and $\varphi_l$ are the spherical harmonics

(3.20) with eigenvalues $l(l+1)$ (for the sphere of the unit radius). One concludes that the operator $\Delta^{(1)}$ on the unit sphere has the eigenvalues $l(l+1)$ with the degeneracy $2(2l+1)$ and $l = 1, 2, \ldots$.

On a side note, we remark that the absence of harmonic vectors on $S^2$ can be understood through the following topological formula

$$n_0 - n_1 + n_2 = \chi_1[\mathscr{M}] \tag{12.100}$$

which relates the Euler characteristic $\chi_1[\mathscr{M}]$, see (1.29), to the so-called Betti numbers $n_p$. The Betti number $B_p$ is the number of harmonic $p$-index antisymmetric fields ($p$-forms). For scalars, $n_0 = 1$ corresponding to one constant harmonic scalar. On a two-dimensional manifold, scalars are Hodge-dual to two-index antisymmetric fields (two-forms), so that $n_2 = 1$ is an expected result. The Euler characteristic of the two-sphere is $\chi_1[S^2] = 2$. Therefore, relation (12.100) predicts $n_1 = 0$, which means the absence of harmonic vector fields.

**3.4** The eigenvalue problem of the Dirac operator is

$$\gamma^\rho \nabla_\rho \psi_\lambda = i\lambda\psi_\lambda. \tag{12.101}$$

The eigenfunctions can be represented as

$$\psi_l = \left[il\varphi_l + \gamma^\mu(\partial_\mu\varphi_l)\right]\epsilon_i, \tag{12.102}$$

$$\psi_{-l-1} = \left[-i(l+1)\varphi_l + \gamma^\mu(\partial_\mu\varphi_l)\right]\epsilon_i, \tag{12.103}$$

in terms of the Killing spinors $\epsilon_i$, see Eq. (12.22), and the eigenfunctions $\varphi_l$ of the scalar Laplacian, $\Delta^{(0)}\varphi_l = l(l+1)\varphi_l$. The index $l$ takes the values $0, 1, 2, \ldots$. The eigenfunctions $\psi_l$ and $\psi_{-l-1}$ correspond to the eigenvalues $il$ and $-i(l+1)$, respectively. One concludes that the spectrum of the Dirac operator on $S^2$ is $\lambda_n = in$, $n = 0 \pm 1, \pm 2, \ldots$. The degeneracy of the modes is $2(2|n|+1)$. Note that the Killing spinors are zero modes of the Dirac operator.

**3.5** One writes

$$(i\gamma^\mu\nabla_\mu)^2\psi = -\gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu\psi = -\frac{1}{2}\{\gamma^\mu\gamma^\nu\}\nabla_\mu\nabla_\nu\psi - \frac{1}{2}\gamma^\mu\gamma^\nu[\nabla_\mu, \nabla_\nu]\psi$$

and uses identities (1.58) for the spin-connection. The rest of the proof follows from the symmetry properties of the Riemann tensor and Clifford relations (1.55).

**3.6** After some algebra Eqs. (3.3) give

$$\omega_\mu = V_\mu + \frac{i}{2}[\gamma_\mu, \gamma_\nu]A^\nu\gamma_*, \tag{12.104}$$

$$E = \frac{1}{4}[\gamma_\mu, \gamma_\nu]F^{\mu\nu} + i\gamma_* D^\mu A_\mu - (n-2)A_\mu A^\mu + \frac{n-3}{4}[\gamma^\mu, \gamma^\nu][A_\mu, A_\nu], \tag{12.105}$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu]$ and $D_\mu A_\nu = \partial_\mu A_\nu + [V_\mu, A_\nu]$. The calculation of the field strength $\Omega_{\mu\nu}$ is a bit lengthy but otherwise straightforward (cf. [243]). It yields

$$\begin{aligned}\Omega_{\mu\nu} &= F_{\mu\nu} - [A_\mu, A_\nu] - i\gamma_*\gamma^\rho(\gamma_\nu D_\mu A_\rho - \gamma_\mu D_\nu A_\rho) + i\gamma_* A_{\mu\nu} \\ &\quad + [A_\mu, A_\rho]\gamma^\rho\gamma_\nu - [A_\nu, A_\rho]\gamma^\rho\gamma_\mu - \not{A}\gamma_\mu\not{A}\gamma_\nu + \not{A}\gamma_\nu\not{A}\gamma_\mu, \end{aligned} \tag{12.106}$$

where $\not{A} = \gamma^\mu A_\mu$ and

$$A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [V_\mu, A_\nu] - [V_\nu, A_\mu]. \tag{12.107}$$

## 12.4 Chapter 4. Heat Equation

**4.1** It is obvious that the Fourier integral

$$K(x, y|t) = (2\pi)^{-n} \int d^n k\, e^{-tk^2} e^{ik_\mu(x-y)^\mu} \tag{12.108}$$

solves heat equation (4.4) for $L = \Delta$ on $\mathbb{R}^n$ and satisfies initial condition (4.5). Next we perform the integral over $k$ to arrive at representation (4.17).

**4.2** Asymptotics of the heat kernel on $S^3$ are easy to get from (4.13). The corresponding eigenvalues and degeneracy are

$$\lambda_l = l(l+2) = (l+1)^2 - 1, \qquad d_l = (l+1)^2, \tag{12.109}$$

see (3.21) and (3.22). Thus, the heat kernel for the Laplacian on $S^3$ reads

$$\begin{aligned} K(\Delta_{S^3}, t) &= \sum_{l=0}^{\infty} d_l e^{-t\lambda_l} = e^t \sum_{m=1}^{\infty} m^2 e^{-tm^2} \\ &= \frac{e^t}{2} \sum_{m=-\infty}^{\infty} m^2 e^{-tm^2} = \frac{e^t}{2}\frac{d}{dt}\left(\sum_{m=-\infty}^{\infty} e^{-tm^2}\right) \\ &= \frac{e^t}{2}\frac{d}{dt}\left(\sqrt{\frac{\pi}{t}} + \mathcal{O}(e^{-1/t})\right) = \frac{e^t}{4t}\sqrt{\frac{\pi}{t}} + \mathcal{O}(e^{-1/t}), \end{aligned} \tag{12.110}$$

where (4.13) has been used.

**4.3** The easiest way to do this calculation is to consider an $(n+1)$-dimensional Gaussian integral

$$\int d^{n+1}x\, e^{-x^2} = \pi^{(n+1)/2}.$$

The same integral calculated in the polar coordinate system reads

$$(\mathrm{vol}\, S^n) \int_0^\infty dr\, r^n e^{-r^2} = (\mathrm{vol}\, S^n)\frac{1}{2}\Gamma\left(\frac{n+1}{2}\right).$$

By comparing this line with the line above one obtains formula (4.123).

**4.4** Consider a two-parameter family of Laplacians, $L(\epsilon_1, \epsilon_2) = e^{-\epsilon_1 f}(L - \epsilon_2 F)$. Due to (4.54)

$$\left.\frac{d}{d\epsilon_1}\right|_{\epsilon_1=0} a_n(1, L(\epsilon_1, \epsilon_2)) = 0.$$

Let us now differentiate this equation w.r.t. $\epsilon_2$

$$\begin{aligned} 0 &= \left.\frac{d}{d\epsilon_2}\right|_{\epsilon_2=0} \left.\frac{d}{d\epsilon_1}\right|_{\epsilon_1=0} a_n(1, L(\epsilon_1, \epsilon_2)) \\ &= \left.\frac{d}{d\epsilon_1}\right|_{\epsilon_1=0} \left.\frac{d}{d\epsilon_2}\right|_{\epsilon_2=0} a_n(1, L(\epsilon_1, \epsilon_2)). \end{aligned} \tag{12.111}$$

Further, by (4.43),

$$\left.\frac{d}{d\epsilon_2}\right|_{\epsilon_2=0} a_n(1, L(\epsilon_1, \epsilon_2)) = a_{n-2}(e^{-\epsilon_1 f} F, L(\epsilon_1, 0)).$$

By substituting this equation in (12.111) one arrives at (4.55).

**4.5** There are three ways to derive (4.125)–(4.127).

Method 1. One can use the variational relation

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} a_k(1, L - \epsilon Q) = a_{k-2}(Q, L), \tag{12.112}$$

which is an analog of (4.43). It is easy to reproduce $a_0(Q, L)$ and $a_2(Q, L)$, but to derive $a_4(Q, L)$ one needs $a_6(1, L)$.

Method 2. One can repeat calculations (4.44)–(4.48) with the replacement $f \to Q$. Since one never commutes any operator through $f$ or $Q$ during these calculations, result (4.125)–(4.127) is obvious.

Method 3. One can find all invariants of proper canonical dimension which may appear in $a_0(Q, L)$, $a_2(Q, L)$ $a_4(Q, L)$. It is easy to understand that only the invariants which appear in (4.125)–(4.127) are allowed. Proper coefficients are recovered by considering the particular case $Q = f \cdot I$, where $I$ is the unit matrix, and by comparing to (4.56)–(4.58). This method also works for curved $\mathscr{M}$.

**4.6** By definition (1.87) we have

$$K_j^j = K_\mu^\mu = -n^\mu{}_{;\mu} = -\partial_\mu n^\mu - \Gamma^\mu_{\rho\mu} n^\rho, \tag{12.113}$$

where we used the identity $n_\mu n^\mu{}_{;\nu} = 0$ which follows from differentiating $n_\mu n^\mu = 1$. It is convenient to suppose that the coordinate system is Gaussian (1.88) before the conformal transformation. Then

$$\delta \partial_\mu n^\mu = -\sigma_{;n}. \tag{12.114}$$

Next, by using (4.51), we obtain

$$\delta \Gamma^\mu_{\rho\mu} n^\rho = \sigma K_j^j + n\sigma_{;n}. \tag{12.115}$$

Then Eq. (4.90) follows from (12.113), (12.114) and (12.115).

**4.7** The method is the following. One rewrites conformal relations (4.97) and (4.98) without using (4.96), i.e. assuming that the constants $\beta_2^{D,N}$ are unknown:

$$-(n-4) f_{;n} - \beta_2^D (n-1) f_{;n} = \beta_3^D (n-2) f_{;n},$$
$$-(n-4) f_{;n} - \beta_2^N (n-1) f_{;n} + 6(n-2) f_{;n} = \beta_3^N (n-2) f_{;n}.$$

Each of these equations should hold for arbitrary $n$ with $\beta$'s independent of $n$. Therefore, the two lines above actually mean four conditions on $\beta$'s, which are enough to get both (4.96) and (4.99).

**4.8** For the symmetric function condition (4.103) is satisfied automatically, while the second matching condition (4.105) yields: $0 = (f^+_{s;n} + f^-_{s;n}) + Vf = 2f_{s;n} + V$, which is the Robin condition with $\mathscr{S} = \frac{1}{2}V$. For the antisymmetric function, continuity condition (4.103) yields the Dirichlet condition. Then the second condition is a consistency check.

**4.9** By assuming that $\Im z > 0$ one finds

$$\sum_{k=-\infty}^{\infty} \frac{e^{ik\alpha}}{z+ka} = -i \sum_{k=-\infty}^{\infty} e^{ik\alpha} \int_0^\infty e^{i(z+ka)\lambda}\, d\lambda$$
$$= -2\pi i \sum_{m=-\infty}^{\infty} \int_0^\infty e^{iz\lambda} \delta(a\lambda + \alpha - 2\pi m)\, d\lambda, \qquad (12.116)$$

where we used (12.88). Let the parameter $a$ be positive. Then for $0 < \alpha < 2\pi$ the contribution to the sum in the right hand side of (12.116) comes from the terms with $m = 1, 2, \dots$ and one immediately arrives at formula (4.129). If $\alpha = 0$ there is an extra contribution to the sum from the term with $m = 0$. By taking into account that on the half axis $\int_0^\infty \delta(\lambda)\, d\lambda = 1/2$ one gets (4.128).

**4.10** The heat kernel with given periodicity property is determined by the summation formula

$$K_{\beta,\alpha}\big(x(\tau), x'(0)|t\big) = \sum_{n=-\infty}^{\infty} e^{in\alpha} K_\infty\big(x(\tau+\beta n), x'(0)|t\big) \qquad (12.117)$$

which is analogous to (4.114). One can use (4.129) to get the following representation:

$$K_{\beta,\alpha}\big(x(\tau), x'(0)|t\big) = \frac{1}{2i\beta} \int_A \frac{e^{i(\pi-\alpha)(z-\tau)/\beta}}{\sin\frac{\pi}{\beta}(z-\tau)} K\big(x(z), x'(0)|t\big)\, dz, \qquad (12.118)$$

which relates $K_{\beta,\alpha}$ with the heat kernel on the plane. The contour $A$ is shown on Fig. 4.1.

**4.11** The proof of formulas (4.131)–(4.132) is based on a residue calculation.

**4.12** We start with the heat kernel $K(x(\tau), x'(0), t)$ of the spinor Laplacian on a two-dimensional plane $\mathbb{R}^2$. Here one can use the Cartesian coordinates and associated veilbein basis. In this case the covariant spinor derivatives are trivial, $\nabla_\mu = \partial_\mu$, and the spinor Laplacian is $\Delta^{(1/2)} = -\partial_\mu^2$. Hence,

$$K\big(x(\tau), x'(0), t\big) = \mathbb{I} K^{(0)}\big(x(\tau), x'(0), t\big), \tag{12.119}$$

where $\mathbb{I}$ is a unit 2 by 2 unit matrix and $K^{(0)}(x(\tau), x'(0), t)$ is the scalar heat kernel (4.109). To proceed it is necessary to go to a veilbein basis associated with the polar coordinates $\tau, \rho$ because the Cartesian basis cannot be used on a cone. The spinor connections for polar coordinates are $\nabla_\mu = \partial_\mu + w_\mu$, $w_\mu\, dx^\mu = -\frac{i}{2}\sigma_3\, d\tau$. Transformation from one sort of connections to another is

$$U(\tau)\partial_\mu U^{-1}(\tau) = \nabla_\mu, \qquad U(\tau) = \exp\left(\frac{i}{2}\sigma_3 \tau\right). \tag{12.120}$$

The heat kernel on $\mathbb{R}^2$ for veilbeins in polar coordinates is $\tilde{K}(x(\tau), x'(0)|t) \equiv U(\tau)K(x(\tau), x'(0)|t)$. The difference between $\tilde{K}$ and $K$ is in the periodicity properties:

$$K\big(x(\tau + 2\pi), x'(0)|t\big) = K\big(x(\tau), x'(0)|t\big),$$

$$\tilde{K}\big(x(\tau + 2\pi), x'(0)|t\big) = -\tilde{K}\big(x(\tau), x'(0)|t\big).$$

The heat kernel on the cone can be defined as

$$K_\beta\big(x(\tau), x'(0)|t\big) = \sum_{n=-\infty}^{\infty} (-1)^n \tilde{K}_\infty\big(x(\tau + \beta n), x'(0)|t\big) \tag{12.121}$$

where

$$\tilde{K}_\infty\big(x(\tau), x'(0)|t\big) \equiv \frac{1}{2\pi i}\int_A \frac{1}{z - \tau}\tilde{K}\big(x(z), x'(0)|t\big)\, dz \tag{12.122}$$

is the spinor heat kernel on the infinitely sheeted Riemann surface ($-\infty < \tau < \infty$). The integration contour $A$ is shown on Fig. 4.1. Kernel (12.121) changes the sign when $\tau$ is increased by $\beta$ and corresponds to the spin connection $w_\mu\, dx^\mu = -\frac{i}{2}\sigma_3\, d\tau$. With the help of (4.129) for $\alpha = \pi$ one then finds the required representation

$$K_\beta\big(x(\tau), x'(0)|t\big) = \frac{1}{2i\beta}\int_A \frac{1}{\sin\frac{\pi}{\beta}(z - \tau)} U(z) K^{(0)}\big(x(z), x'(0)|t\big)\, dz. \tag{12.123}$$

This is an analog of the Sommerfeld formula for the spinor Laplacian.

To study the asymptotic expansions, Eq. (12.123) can be written as

$$\begin{aligned} &K_\beta\big(x(\tau), x'(0)|t\big) \\ &\quad = K\big(x(\tau), x'(0)|t\big) + \frac{1}{2i\beta}\int_{A'} \frac{1}{\sin\frac{\pi}{\beta}(z - \tau)} U(z) K\big(x(z), x'(0)|t\big)\, dz. \end{aligned} \tag{12.124}$$

The contour $A'$ consists of two vertical lines. The smeared trace $K_\beta(f,L;t)$ with a test function $f$ which does not depend on $\tau$ can be written as

$$K_\beta(f,L;t)$$
$$= t^{-1}a_0(f,L) + \frac{1}{2i}\int_{A'} \frac{\operatorname{tr} U(z)}{\sin\frac{\pi}{\beta}z} \int_0^\infty \frac{\exp(-\frac{\rho^2}{t}\sin^2\frac{z}{2})}{4\pi t} f(\rho)\rho\, d\rho\, dz, \quad (12.125)$$

$$a_0(f,L) = \frac{1}{4\pi}\int_{C_\beta} d^2x\,\sqrt{g}\,f(x)\operatorname{tr}\mathbb{I}. \quad (12.126)$$

One can note that $\operatorname{tr} U(z) = \cos(z/2)\operatorname{tr}\mathbb{I}$. By taking into account formula (4.132) and acting in the same way as in Sect. 4.7 we arrive at the following result:

$$K_\beta(f,L;t) \sim t^{-1}a_0(f,L) + a_2(f,L) + \cdots, \quad (12.127)$$

$$a_2(f,L) = -\frac{1}{24\gamma}(\gamma^2-1)f(0)\operatorname{tr}\mathbb{I}, \quad (12.128)$$

where $\gamma = \frac{2\pi}{\beta}$.

## 12.5 Chapter 5. Spectral Functions

**5.1** The domain of convergence of the series follows from explicit expressions (3.21), (3.22) for the spectrum of the Laplacian on an $n$ sphere.

**5.2** One can note that the heat kernel for the Laplacian on $S^2$ is represented as

$$K(\Delta;t) = \sum_{l=0}^{\infty}(2l+1)e^{-tl(l+1)} = \sum_{l=0}^{\infty}(2l+1)e^{-t(l+1/2)^2}e^{t/4}$$
$$= e^{t/4}K(\Delta+1/4;t). \quad (12.129)$$

The operator $\Delta + 1/4$ on $S^2$ is positive definite and its zeta-function is related to the generalized Riemann zeta-function (5.4),

$$\zeta(s,\Delta+1/4) = \sum_{l=0}^{\infty}(2l+1)(l+1/2)^{-2s} = 2\zeta_R(2s-1,1/2). \quad (12.130)$$

According to (5.28) the heat kernel coefficients are expressed through the poles of $\Gamma(s)\zeta(s)$,

$$a_p(\Delta+1/4) = 2\operatorname{Res}_{s=(2-p)/2}(\Gamma(s)\zeta_R(2s-1,1/2)). \quad (12.131)$$

As we have explained in Sect. 5.2 the zeta-function $\zeta_R(z,a)$ has a single pole at $z=1$ with a unit residue. Consequently, $\zeta_R(2s-1,1/2)$ has a pole at $s=1$ with Res $= \frac{1}{2}$. The corresponding heat-kernel coefficient is $a_0(\Delta+1/4) = 1$. The poles of $\Gamma(s)$ at non-positive integers $s$ generate higher heat kernel coefficients. The residues are defined by (5.20), (5.14) and (5.15). In particular, $a_2(\Delta + 1/4) = \frac{1}{12}$, $a_4(\Delta + 1/4) = \frac{7}{32\cdot 15}$.

Thus, we find that

$$K(\Delta; t) \simeq e^{t/4}\left(\frac{1}{t} + \frac{1}{12} + \frac{7t}{32\cdot 15} + \mathcal{O}(t^2)\right) \tag{12.132}$$

in accord with Eq. (4.15).

**5.3** The spectrum of single-particle energies in this model is $\omega_n = |k_n|$, $k_n = \frac{2\pi}{l}n + \frac{b}{l}$, $n = 0, \pm 1, \pm 2, \dots$. In the $\zeta$-function regularization

$$\begin{aligned} E_0 &= \frac{2\pi}{l}\left[\zeta\left(-1, \frac{b}{2\pi}\right) + \zeta\left(-1, -\frac{b}{2\pi}\right) + \frac{b}{2\pi}\right] \\ &= -\frac{2\pi}{l}\left(\left(\frac{b}{2\pi}\right)^2 + \frac{1}{6}\right) + \frac{b}{l}. \end{aligned} \tag{12.133}$$

**5.4** If we assume some regularization of the singularity in $S_\beta$, the spheres $S^2$ and $S^2_\beta$ have the same topologies and the same Hodge-de Rham decompositions. One concludes basing on the results of Exercise 3.3 that in the both cases the relation between the heat kernels of the vector and scalar Laplacians has the form

$$\mathrm{Tr}\, K(\Delta^{(1)}; t) - n_1 = 2\, \mathrm{Tr}\, K(\Delta^{(0)}; t) - 2n_0. \tag{12.134}$$

Here $n_1$ and $n_0$ are the numbers of zero modes of the operators $\Delta^{(1)}$ and $\Delta^{(0)}$, respectively. These numbers do not depend on either the base manifold is $S^2$ or $S^2_\beta$, $n_0 = 1$, $n_1 = 0$. Therefore,

$$a_2(\Delta^{(1)}) = 2a_2(\Delta^{(0)}) - 2. \tag{12.135}$$

The form of the scalar coefficient $a_2(\Delta^{(0)})$ on $S^2_\beta$ is known from results of Sect. 4.7. It is not difficult conclude from (12.135) that on a cone

$$a_2(\Delta^{(1)}) = 2a_2(\Delta^{(0)}) + (\gamma^{-1} - 1) \tag{12.136}$$

where $\gamma = 2\pi/\beta$. The latter formula agrees with (4.122) for $n = 2$.

**5.5** We discuss in details theory (5.99) in three-dimensional spacetime, $n = 2$. Generalization to $n = 3$ is straightforward, see discussion in [101].

The spatial components of the vector-potential can be chosen as $A_1 = 0$, $A_2 = Bx^1$, where $B$ is the strength of the magnetic field. Laplacian (5.100) takes the form

$$L(A) = -\partial_x^2 + (\partial_y + ieBx)^2 + m^2. \tag{12.137}$$

For the large volume the eigenfunctions of this operator are

$$\varphi_\Lambda(x, y) = \frac{1}{\sqrt{2\pi}} e^{ip_k y}\psi_l(x),$$

where $\psi_n(x)$ are the eigenfunctions for the problem

$$\left(-\partial_x^2 - (p_k + eBx)^2 + m^2\right)\psi_l(x) = \lambda_l \psi_l(x). \qquad (12.138)$$

After a coordinate redefinition, $x' = x + p_k/(eB)$, one reduces (12.138) to an eigenvalue problem which determines the spectrum of a harmonic oscillator with the frequency $\omega = eB$ in quantum mechanics (we assume that $eB$ is a positive constant). One easily concludes that

$$\lambda_l = 2eB\left(l + \frac{1}{2}\right) + m^2,$$

where $l = 0, 1, 2, \dots$.

The summation over $p_k$ in the spectral functions yields a factor $C(V, B)$. This factor is formally divergent even if $V$ is finite because eigenvalues of $L(A)$ coincide with $\lambda_n$. The divergence occurs because finite size effects were ignored. If $L$ is a size of the system along the $x$ coordinate the eigenvalues of the operator $L(A)$ should coincide with those in the absence of the gauge field at $|p_k| \gg BeL$. To fix $C(V, B)$ consider the trace of the heat kernel

$$\mathrm{Tr}\, e^{-tL(A)} = C(V, B)\sum_{l=0}^{\infty} e^{-t\lambda_l} = \frac{C(V, B)}{\cosh(Bet)} e^{-tm^2}. \qquad (12.139)$$

The short $t$ asymptotic of this trace is determined by large eigenvalues and it should coincide with the asymptotic of the operator when the gauge field is absent,

$$\mathrm{Tr}\, e^{-tL(A)} \sim \frac{2V}{4\pi t}. \qquad (12.140)$$

(Factor 2 appears because the scalar field is complex.) By comparing (12.139) and (12.140) one concludes that $C(V, B) = Be/(2\pi)$. For the zeta-function of $L(A)$ one, therefore, has

$$\zeta(s; L(A)) = \frac{BeV}{2\pi}\sum_{l=0}^{\infty}\left(2eB\left(l + \frac{1}{2}\right) + m^2\right)^{-s} = \frac{V(2Be)^{1-s}}{4\pi}\zeta_R(s, a),$$
$$a = \frac{1}{2} + \frac{m^2}{2eB}, \qquad (12.141)$$

where $\zeta_R(s, a)$ is the generalized Riemann zeta-function (5.4).

This solution is taken from [39].

**5.6** By definition, the asymptotic expansion of the trace derivative yields

$$\frac{d}{d\alpha}\mathrm{Tr}\, e^{-tL_\alpha} \sim \sum_{p=0} t^{(p-n)/2}\frac{d}{d\alpha} a_p(L_\alpha). \qquad (12.142)$$

It follows from (5.63) that

$$\frac{d}{d\alpha}\mathrm{Tr}\, e^{-tL_\alpha} = -t\,\mathrm{Tr}\left(\mathcal{O} L_\alpha e^{-tL_\alpha}\right) = t\frac{d}{dt}\mathrm{Tr}\left(\mathcal{O}\, e^{-tL_\alpha}\right). \qquad (12.143)$$

This yields (5.101).

**5.7** Suppose $\varphi_1^{(0)}$ is a zero mode of $D_+$. Then $L_1\varphi_1^{(0)} = D_- D_+ \varphi_1^{(0)} = 0$, and $\varphi_1^{(0)}$ is a zero eigenmode of $L_1$. Suppose that $\tilde{\varphi}_1^{(0)}$ is a zero eigenmode of $L_1$. Then, $0 = (\tilde{\varphi}_1^{(0)}, L_1\tilde{\varphi}_1^{(0)})_1 = (D_+\tilde{\varphi}_1^{(0)}, D_+\tilde{\varphi}_1^{(0)})_2$. Since the inner product $(.,.)_2$ is positive definite, $D_+\tilde{\varphi}_1^{(0)} = 0$. This completes the proof.

**5.8** Let $L_1\varphi_1^{(\lambda)} = \lambda\varphi_1^{(\lambda)}$. Consider $\varphi_2^{(\lambda)} \equiv D_+\varphi_1^{(\lambda)}$. Then $L_2\varphi_2^{(\lambda)} = L_2 D_+ \varphi_1^{(\lambda)} = D_+ L_1 \varphi_1^{(\lambda)} = \lambda D_+ \varphi_1^{(\lambda)} = \lambda \varphi_2^{(\lambda)}$, so that $\varphi_2^{(\lambda)}$ is an eigenmode of $L_2$ with the same eigenvalue $\lambda$. This argumentation fails only if $\varphi_2^{(\lambda)} = D_+\varphi_1^{(\lambda)} = 0$, i.e., if $\varphi_1^{(\lambda)}$ is a zero eigenmode of $L_1$. One can also repeat the same calculations after exchanging the roles of $L_1$ and $L_2$.

**5.9** Equation (5.102) implies the following transformation of the operator:

$$\not{D}(B') = U^+ \not{D}(B) U, \tag{12.144}$$

or, in infinitesimal form,

$$\delta \not{D}(B) = i(\not{D}(B)\lambda - \lambda \not{D}(B)), \tag{12.145}$$

where $\lambda^+ = \lambda$ belong to the Lie algebra of $SU(N)$. The chiral parts $D_\pm(B)$ of the Dirac operator transform in the same way,

$$\delta D_\pm(B) = i(D_\pm(B)\lambda - \lambda D_\pm(B)). \tag{12.146}$$

Consider transformation of the phase. Equations (5.89), (5.92) and (5.97) yield

$$\delta\Phi(\hat{D}(B)) = -\zeta(0, \not{D}^2(B), \gamma_\star\lambda). \tag{12.147}$$

Note that (5.97) shows that transformation (12.144) does not change the absolute value of $\det \hat{D}(B)$. If zero modes are neglected one finds after simple algebra and with the help of (4.126) for $n = 2$

$$\delta\Phi(\hat{D}(B)) = -\frac{i}{4\pi}\int_{\mathscr{M}} d^2x\, \widetilde{\mathrm{tr}}(\lambda F), \tag{12.148}$$

where $F = \varepsilon^{\mu\nu}F_{\mu\nu}$, $F_{\mu\nu} = [D_\mu(B), D_\nu(B)]$, $F^+ = -F$, and the trace $\widetilde{\mathrm{tr}}$ extends over the indices of the gauge group. For $n = 4$ it follows from (4.127) that

$$\delta\Phi(\hat{D}(B)) = -\frac{1}{32\pi^2}\int_{\mathscr{M}} d^4x\, \widetilde{\mathrm{tr}}(\lambda F_{\mu\nu}\tilde{F}^{\mu\nu}), \tag{12.149}$$

where $\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$. To derive (12.148), (12.149) we used the representation $\not{D}^2(B) = -(D_\mu(B)D^\mu(B) + E)$, $E = -\frac{1}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}$.

**5.10** First of all, from Eqs. (5.96), (5.98) one finds

$$\delta_\pm \ln\det\not{D} = 2\delta_\pm \ln|\det\hat{D}|_{\bar{D}=D_-} = \frac{1}{2}\zeta\left(0, \not{D}^2, (\mathscr{O}_\pm + \mathscr{O}_\pm^+)\right). \tag{12.150}$$

Let us reproduce (12.150) by using Eq. (5.71),

$$\delta_\pm \ln\det \not{D} = \frac{1}{2}\delta_\pm \ln\det L = \frac{1}{2}\zeta(0, L, \tilde{\mathscr{O}}_\pm), \qquad (12.151)$$

where $L = \not{D}^2$ and generators $\tilde{\mathscr{O}}_\pm$ should be determined from (5.64). One has

$$\begin{aligned}\delta_\pm \operatorname{Tr} e^{-tL} &= 2\delta_\pm \operatorname{Tr} e^{-tD_+D_-} \\ &= -2t\delta_\pm \operatorname{Tr}\big[(\delta_\pm D_+)D_- + D_+(\delta_\pm D_-)\big]e^{-tD_+D_-} \\ &= -2t \operatorname{Tr}(\mathscr{G} + \mathscr{G}^+)\big(D_+D_-e^{-tD_+D_-} \pm D_-D_+e^{-tD_-D_+}\big), \end{aligned} \qquad (12.152)$$

where we have used the arguments given for (5.90). By comparing (12.152) with (5.64) one concludes that

$$\tilde{\mathscr{O}}_+ = 2(\mathscr{G} + \mathscr{G}^+), \qquad \tilde{\mathscr{O}}_- = -2\gamma_\star(\mathscr{G} + \mathscr{G}^+). \qquad (12.153)$$

Thus, $\tilde{\mathscr{O}}_\pm = \mathscr{O}_\pm + \mathscr{O}_\pm^+$, see (5.92), and (12.151) coincides with (12.150).

## 12.6 Chapter 6. Non-linear Spectral Problems

**6.1** We start with definition of product for two eigenfunctions

$$\begin{aligned}(\omega^2 - (L_2 + \omega L_1 + \omega^2 L_0))\phi_\omega = 0, \\ (\sigma^2 - (L_2 + \sigma L_1 + \sigma^2 L_0))\psi_\sigma = 0. \end{aligned} \qquad (12.154)$$

Suppose first that $\omega \neq \sigma$. Then

$$\begin{aligned}\langle \phi_\omega, \psi_\sigma \rangle &= (\omega + \sigma)(\phi_\omega, (1 - L_0)\psi_\sigma) - (\phi_\omega, L_1\psi_\sigma) \\ &= \frac{1}{(\omega - \sigma)}(\phi_\omega, ((\omega^2 - \sigma^2)(1 - L_0) - (\omega - \sigma)L_1)\psi_\sigma) \equiv 0, \end{aligned} \qquad (12.155)$$

where we used Eqs. (12.154).

If $\omega = \sigma$ (12.155) becomes

$$\langle \phi_\omega, \psi_\omega \rangle = (\phi_\omega, (2\omega(1 - L_0) - L_1)\psi_\omega) = (\phi_\omega, (2\omega - \partial_\omega L(\omega))\psi_\omega), \qquad (12.156)$$

where $\partial_\omega L(\omega) = L_1 + 2\omega L_0$. Let us take into account that $\psi_\omega$ coincides with an eigenfunction $\phi_\Lambda^{(\omega)}$ of an operator $L(\omega)$ with some eigenvalue $\Lambda(\omega)$,

$$L(\omega)\phi_\Lambda^{(\omega)} = \Lambda(\omega)\phi_\Lambda^{(\omega)} \qquad (12.157)$$

(as earlier, we imply that $\Lambda(\omega)$ is specified by some indexes but do not write them explicitly). The parameter $\omega$ is determined from the equation $\omega^2 = \Lambda(\omega)$. Let $\omega$ in (12.157) be a free real parameter. Take the derivative of the both sides of (12.157) over $\omega$ and in the final formula put $\omega^2 = \Lambda(\omega)$. We get

$$\partial_\omega L(\omega)\psi_\omega = \partial_\omega \Lambda(\omega)\psi_\omega + (\omega^2 - L(\omega))\xi_\omega, \qquad (12.158)$$

where $\xi_\omega = \partial_\omega \phi_\Lambda^{(\omega)}$ at $\omega^2 = \Lambda(\omega)$. Substitution of (12.158) in (12.156) yields

$$\langle \phi_\omega, \psi_\omega \rangle = (2\omega - \partial_\omega \Lambda(\omega))(\phi_\omega, \psi_\omega), \qquad (12.159)$$

which coincides with (6.14).

**6.2** The pseudo-trace corresponding to (6.58) is

$$K(t) = \frac{1}{2}\sum_{\omega} e^{-t\omega^2} = \frac{1}{2}\sum_{\lambda}\left(e^{-t(\sqrt{\lambda}-\varrho)^2} + e^{-t(\sqrt{\lambda}+\varrho)^2}\right), \qquad (12.160)$$

where $\lambda$ are eigenvalues of $L_2$. One can consider (12.160) as a result of a non-linear transformation of the spectrum, $\lambda \to (\sqrt{\lambda} \pm \varrho)^2$. Expression (12.160) can be rewritten as

$$K(t) = e^{-t\rho^2}\sum_{\lambda} e^{-t\lambda}\cosh(2t\rho\sqrt{\lambda}). \qquad (12.161)$$

By using Taylor series for $\cosh x$ one can represent (12.161) in the following form:

$$K(t) = e^{-t\rho^2}\sum_{p=0}^{\infty}\frac{(t\rho)^{2p}}{(2p)!}(-1)^p\partial_t^p K(L_2;t). \qquad (12.162)$$

This expression can be used to study short $t$ expansions of $K(t)$ starting from the asymptotic $K(L_2;t)$. The check of (6.20), (6.21) is now straightforward.

**6.3** Equation (6.3) can be written as

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{2s-1}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(n-|m|+1)e^{-(n+1+m\Omega)t}\,dt. \qquad (12.163)$$

Summation over $m$ and $n$ can be perform explicitly. After that one should study the poles of the integral at $s = 1/2$ and $s = 3/2$ and use (6.61), (6.62). This again can be done explicitly because the poles appear as a result of the divergence of the integral on the lower limit.

## 12.7 Chapter 7. Effective Action

**7.1** Let $\varphi$ be a function from the Hilbert space of the operator $L$, and let $\{\varphi_\lambda\}$ be a set of normalized eigenfunctions of $L$ with eigenvalues $\lambda$. One has

$$\varphi = \sum_{\lambda} b_\lambda \varphi_\lambda, \qquad b_\lambda = (\varphi_\lambda, \varphi). \qquad (12.164)$$

The "integration" in (7.131) can be formally specified as a usual integration over the coefficients $b_\lambda$,

$$\int [D\varphi]e^{-(\varphi,L\varphi)} = \prod_{\lambda}\int_{-\infty}^{\infty} db_\lambda\, e^{-\lambda|b_\lambda|^2}. \qquad (12.165)$$

This expression has a meaning if the spectrum of $L$ is restricted, $\lambda \leq \Lambda$, by a ultra-violet cutoff $\Lambda$. Then

$$\int [D\varphi]e^{-(\varphi,L\varphi)} = N(\Lambda)\exp\left(-\frac{1}{2}\sum_{\lambda\leq\Lambda}\ln\lambda\right), \qquad (12.166)$$

where $N(\Lambda)$ is a numerical coefficient. Finally one can replace the series $\sum_{\lambda\leq\Lambda}\ln\lambda$ by the Ray-Singer formula and take the limit $\Lambda \to \infty$.

**7.2** One uses eigenfunctions of $\not{D}$,

$$\not{D}\psi_\lambda = \lambda\psi_\lambda.$$

The eigenfunctions of the conjugated operator are $\bar{\psi}_\lambda$. The eigenfunctions are "normalized", $(\bar{\psi}_\lambda, \psi_\sigma) = \delta_{\lambda\sigma}$. The field allows a decomposition

$$\psi(x) = \sum_\lambda c_\lambda \psi_\lambda(x), \qquad \bar{\psi}(x) = \sum_\lambda \bar{c}_\lambda \bar{\psi}_\lambda(x), \tag{12.167}$$

where $c_\lambda$, $\bar{c}_\lambda$ are some Grassman variables. By taking into account (7.37), (7.35) one should adopt the integration rules

$$\int dc_\lambda = 0, \quad \int d\bar{c}_\lambda = 0, \quad \int \bar{c}_\lambda\, dc_\sigma = \int c_\lambda\, d\bar{c}_\sigma = 0, \tag{12.168}$$

$$\int c_\lambda\, dc_\sigma = \delta_{\lambda\sigma}, \quad \int \bar{c}_\lambda\, d\bar{c}_\sigma = \delta_{\lambda\sigma}. \tag{12.169}$$

The integral (7.132) can be formally written as

$$\int [D\bar{\psi}][D\psi] e^{-(\bar{\psi}, \not{D}\psi)} = \prod_\lambda \int d\bar{c}_\lambda\, dc_\lambda e^{-\lambda \bar{c}_\lambda c_\lambda}. \tag{12.170}$$

This yields, in analogy with the Bose case,

$$\int [D\bar{\psi}][D\psi] e^{-(\bar{\psi}, \not{D}\psi)} = \exp\left( \sum_{0<\lambda\le\Lambda} \ln\lambda + \sum_{-\Lambda\le\lambda<0} (\ln(-\lambda) + i\pi) \right), \tag{12.171}$$

where $\Lambda$ is a cutoff. The quantity in the exponent can be related to first order derivative of the zeta-function of the Dirac operator (5.56), see Sect. 5.6. Then one can use formulas (5.57)–(5.59) and remove the cutoff $\Lambda$.

**7.3** Let us represent (7.9) as

$$Z(\beta) = \int dE\, \mu(E) e^{-\beta E}, \tag{12.172}$$

$$\mu(E) = \sum_{n=0}^{\infty} \delta(E - E_n). \tag{12.173}$$

Function $\mu(E)$ is the spectral density of the energy operator. One can put $\mu(E) = e^{f(E)}$ and estimate the integral in (12.172) as

$$Z(\beta) \simeq e^{f(E_*) - \beta E_*}, \tag{12.174}$$

where $E_* = E_*(\beta)$ is defined by the condition

$$\partial_E f(E)_{E=E_*} = \beta. \tag{12.175}$$

It is easy to see by using (7.13), (7.15) that in the given approximation

$$\mathscr{E}(\beta) = E_*(\beta), \tag{12.176}$$

$$S(\beta) = f(E_*(\beta)). \tag{12.177}$$

Therefore $\mu(E) \simeq e^S$ and the number of states in the interval of energies $(E, E + \Delta E)$ is $e^S \Delta E$.

**7.4** The single-particle spectrum for such theory in $n = d + 1$ dimensional space-time is

$$\omega(n_1, \dots, n_d) = (k_1^2 + k_2^2 + \cdots + k_d^2)^{1/2}. \tag{12.178}$$

where $k_p = \pi n_p / l$ and $n_p = 1, 2, \dots$. The free energy

$$F(\beta) = \beta^{-1} \sum_{n_p} \ln\bigl(1 - e^{-\omega(n_1,\dots,n_d)}\bigr) \tag{12.179}$$

in the thermodynamical limit is reduced to the integral

$$\begin{aligned} F(\beta) &= \beta^{-n} \frac{V}{\pi^d} \int_{x_i > 0} d^d x \ln\bigl(1 - e^{-x_i^2}\bigr) \\ &= -T^n \frac{V}{d(2\pi)^d} \Gamma(n)\zeta(n)\Omega(S^{d-1}), \end{aligned} \tag{12.180}$$

where $V = l^d$ and $\Omega(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of the hypersphere $S^{d-1}$. The final answer is

$$F(\beta) = -T^n \frac{V}{\pi^{d/2}} \Gamma\left(\frac{n}{2}\right) \zeta(n). \tag{12.181}$$

It is not difficult to get from (12.181) and definition (7.13) the Stefan-Boltzmann law (7.134).

**7.5** The free energy of a scalar field on an interval of length $l$ with the Dirichlet boundary conditions is determined by the single-particle spectrum $\omega_k = \pi k/l$, where $k = 1, 2, \dots$,

$$F(\beta) = \beta^{-1} \sum_{k=1} \ln\bigl(1 - e^{-\pi\beta k/l}\bigr). \tag{12.182}$$

In the thermodynamical limit, $\beta/l \ll 1$, one can replace the sum by an integral and use the Euler-MacLourain formula

$$\sum_{n=1}^{\infty} f(n) = \int_0^{\infty} dx\, f(x) - \int_0^1 dx\, f(x) + \frac{1}{2} f(1) + \sum_{k=1}^{\infty} (-1)^k \frac{B_{k+1}}{(k+1)!} f^{(k)}(1), \tag{12.183}$$

where $B_k$ are the Bernoulli numbers and a function $f(x)$ is supposed to decrease at infinity together with all its derivatives. With help of (12.183) one gets

$$F(\beta) \simeq -\frac{l\pi}{6\beta^2} - \frac{1}{2\beta} \ln \frac{\beta}{2l}. \tag{12.184}$$

The leading term reproduces result (12.181).

**7.6** To define the free energy one needs the single-particle spectrum in the rotating frame. The spectrum in the frame which is at rest is $\omega_k = 2\pi k/l$, $k = 1, 2, \ldots$. The quantity $p_k = \pm 2\pi k/l$ is the momentum of a particle rotating clock-wise or counter-clock-wise, the angular momentum of the particle is $m_k = l p_k/(2\pi)$ (taking into account that $l/(2\pi)$ is the radius of the circle). Thus, in the frame which rotates with the angular velocity $\Omega$ the single-particle spectrum is

$$\omega_k^{\pm}(\Omega) = \omega_k \pm \Omega k = (1 \pm \bar{\Omega})\omega_k, \tag{12.185}$$

where $\bar{\Omega} = l\Omega/(2\pi)$. The partition function is

$$F(\beta) = F_+(\beta) + F_-(\beta), \tag{12.186}$$

where $F_\pm(\beta)$ have the same form as partition function for the system on an interval with the length $l_\pm = l/(1 \pm \bar{\Omega})$. One can now apply formula (12.181) to get

$$F(\beta) \simeq -\frac{l\pi}{6\beta(1-\bar{\Omega}^2)}. \tag{12.187}$$

**7.7** We focus on the proof of (7.26), the case of Bose statistics. Generalization to Fermi statistics is straightforward. One should use the relation

$$\begin{aligned}\sum_{l=1}^{\infty} \ln\left(1 + \frac{\omega^2}{\sigma_l^2}\right) &= \lim_{N\to\infty}\left[\sum_{l=1}^{N} \ln(\sigma_l^2 + \omega^2) - \sum_{l=1}^{N} \ln(\sigma_l^2)\right] \\ &= -\lim_{s\to 0}\frac{d}{ds}\left[\sum_{l=1}^{\infty}(\sigma_l^2+\omega^2)^{-s} - \sum_{l=1}^{\infty}(\omega^2)^{-s}\right]\end{aligned} \tag{12.188}$$

and note that

$$\lim_{s\to 0}\frac{d}{ds}\sum_{l=1}^{\infty}(\omega^2)^{-s} = -\ln\beta. \tag{12.189}$$

Relation (7.26) follows from (7.135), (12.188) and (12.189).

**7.8** We assume that a background manifold is closed and prove (7.137). Generalization to spaces with boundaries is straightforward. One writes

$$\mathrm{Tr}\, e^{-tP_E} = \sum_{l=-\infty}^{\infty} e^{-\sigma_l^2 t}\, \mathrm{Tr}\, e^{-tL(i\sigma_l)}, \tag{12.190}$$

where $\sigma_l$ are defined in (7.25), and uses (6.18) and (6.19) to get the following short $t$ expansion:

$$\mathrm{Tr}\, e^{-tP_E} \sim \sum_{p=0}^{\infty} t^{p-(n-1)/2} \sum_{q=0}^{p} a_{q,p} \sum_{l=-\infty}^{\infty} e^{-\sigma_l^2 t}\sigma_l^q. \tag{12.191}$$

In the limit of small $t$ for $m$ even one has the asymptotic formula

$$\sum_{l=-\infty}^{\infty} e^{-\sigma_l^2 t}\sigma_l^q \sim \frac{\beta}{2\pi}\Gamma\left(\frac{q+1}{2}\right)t^{-(q+1)/2}, \tag{12.192}$$

for $m$ odd the sum vanishes. By using (12.192) in (12.191) and comparing the result with the asymptotic of $K(t;P_E)$ one gets (7.137).

**7.9** Formula (7.139) follows from (7.137) and (6.20). (One should change $n$ to $n-1$ in (6.20).)

**7.10** To avoid the problem of divergences one can modify (7.52) by introduction, for example, a factor $e^{i\epsilon z}$,

$$f_1(s) = -\frac{\varrho^{-2s}}{4\pi}\sum_\lambda \int_{C_+} dz\left[(z^2+\lambda(iz))^{-s} + (z^2+\lambda(-iz))^{-s}\right]e^{i\epsilon z}, \tag{12.193}$$

where $\epsilon > 0$ is a small parameter which serves to regularize the integral near $s=0$. One can integrate in (12.193) by parts and neglect terms which are linear in $\epsilon$. This yields

$$f_1(s) = -s\frac{\varrho^{-2s}}{4\pi}\sum_\omega \int_{C_+} dz\, z\left[\frac{\partial_z\breve{\chi}(z,\lambda)}{(z^2+\lambda(iz))^{s+1}} + \frac{\partial_z\breve{\chi}(-z,\lambda)}{(z^2+\lambda(-iz))^{s+1}}\right]e^{i\epsilon z}. \tag{12.194}$$

The presence of the regularizing factor enables one to replace $C_+$ with a closed contour lying in the upper part of the complex plane and use the Cauchy theorem to get:

$$E_0 = \frac{1}{2}\sum_i \omega_i e^{-\epsilon\omega_i}, \tag{12.195}$$

where $\omega_i$ are single-particle frequencies. Derivation of (12.195) is analogous to the derivation of Eq. (7.56). The quantity (12.195) is the regularized vacuum energy, where the contribution of high-frequency modes is suppressed at the scale $\omega \sim \epsilon^{-1}$.

**7.11** On a closed base manifold the singular part of the zeta-function is, see Eq. (5.28),

$$\zeta^{(\mathrm{pole})}(s;P_E) = \frac{2}{\Gamma(s)}\sum_{k=0}^{\infty}\frac{a_{2k}(P_E)}{2s+2k-n}, \tag{12.196}$$

where $a_{2k}(P_E)$ are the heat coefficients of $P_E$. To investigate the poles of $f_1(s)$ we rewrite (7.52) as

$$\begin{aligned} f_1(s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\,\zeta\big(s;(L(ix)+x^2)\big) \\ &= \frac{1}{2\pi\Gamma(s)}\int_{-\infty}^{\infty} dx\int_0^\infty dt\, t^{s-1}\,\mathrm{Tr}\, e^{-t(L(ix)+x^2)}. \end{aligned} \tag{12.197}$$

We put here $\varrho = 1$, for simplicity. The operator $L(x)$ is defined in (7.41). The poles of $f_1(s)$ are determined by the behaviour of the integral at small $t$. Therefore, one can define the pole part $f_1^{(\text{pole})}(s)$ of $f_1(s)$ by (12.197) where the integration over $t$ is restricted by the interval $(0, 1)$. The heat kernel of $L(ix)$ can be replaced here by its asymptotic (6.18)

$$f_1^{(\text{pole})}(s) = \frac{1}{2\pi\Gamma(s)} \int_{-\infty}^{\infty} dx \int_0^1 dt\, t^{s-1} e^{-tx^2} \sum_{k=0}^{\infty} a_{2k}(L(ix)) t^{(2k-n+1)/2}. \tag{12.198}$$

The integral exists at $\Re\, s > (n-1)/2$. One can now use (6.18) and the left formula in (6.19). After integration in (12.198) over $x$ and $t$ one gets

$$f_1^{(\text{pole})}(s) = \frac{1}{\pi\Gamma(s)} \sum_{k=0}^{\infty} \sum_{m=0}^{[k/2]} \frac{\Gamma(m+1/2)}{(2s+2(k-m)-n)} (-1)^m a_{2m,k}. \tag{12.199}$$

Note that odd powers of $x$ do not contribute to (12.199), while each even power of the order $2m$ yields the factor $i^{2m} = (-1)^m$. The result can be rewritten as

$$f_1^{(\text{pole})}(s) = \frac{1}{\pi\Gamma(s)} \sum_{k=0}^{\infty} \frac{1}{2s+2k-n} \sum_{m=k}^{2k} \Gamma(m-k+1/2)(-1)^{m-k} a_{2(m-k),m}. \tag{12.200}$$

One can now use (7.137) to get (12.196) from (12.200).

**7.12** A straightforward computation yields

$$\begin{aligned} G_\beta^+(x,x') = &\sum_i \frac{1}{1-e^{-\beta\omega_i^+}} f_i^{(+)}(x)(f_i^{(+)}(x'))^* \\ &+ \sum_j \frac{1}{e^{\beta\omega_j^-}-1} (f_j^{(-)}(x'))^* f_j^{(-)}(x), \end{aligned} \tag{12.201}$$

$$\begin{aligned} G_\beta^-(x,x') = &\sum_j \frac{1}{1-e^{-\beta\omega_i^-}} (f_j^{(-)}(x'))^* f_j^{(-)}(x) \\ &+ \sum_i \frac{1}{e^{\beta\omega_j^+}-1} f_i^{(+)}(x)(f_i^{(+)}(x'))^*. \end{aligned} \tag{12.202}$$

At the vanishing temperature, $\beta^{-1} = 0$, expressions (12.201), (12.202) coincide with (2.56), (2.57).

To study the properties of the Wightman functions under analytical continuation in the complex plane of the time coordinate we first note that the Green's functions depend only on the difference in time coordinates of the two points. It is convenient to separate time and spatial coordinates and introduce the definition

$$G_\beta^\pm(t; y, y') = G_\beta^\pm(x, x'), \tag{12.203}$$

for space-time coordinates $x = (t, y^k)$, $x' = (0, (y')^k)$ with $y^k$ being purely spatial coordinates. Consider an analytical continuation of the Wightman function in the complex plane. It is easy to see that $G^+_\beta(z; y, y')$ where $z = t + i\tau$ can be defined in the strip $-\beta < \tau < 0$. This fact follows from (12.201), the positivity of the single-particle spectrum, and the definition of the single-particle modes, which implies that

$$f_i^{(\pm)}(t + i\tau, y) = e^{\pm\tau\omega_i^{(\pm)}} f_i^{(\pm)}(t, y). \tag{12.204}$$

If $-\beta < \tau < 0$ the series (12.201) for $G^+_\beta(t+i\tau; y, y')$ converge at large $\omega_i^{(\pm)}$. Analogously, it follows from (12.202) that $G^-_\beta(z; y, y')$ with $z = t + i\tau$ can be defined in the strip $0 < \tau < \beta$, $-\infty < t < \infty$.

**7.13** To demonstrate that relations (7.140), (7.141) define a single function in some domain of the complex plane, let us note that operators $\varphi(x)$ and $\varphi^+(x')$ commute if points $x$ and $x'$ are taken on a Cauchy hypersurface in a globally hyperbolic space-time, see Exercise 2.4. We are working with stationary space-times implying the global hyperbolicity. The constant time hypersurfaces are an example of the Cauchy hypersurfaces.

We assume that a constant time Cauchy hypersurface allows at least small deformations. This means that two casually independent points $x = (t, y)$ and $x' = (0, y')$ may be on a Cauchy hypersurface at least for some values of $t$. The commutator of operators at these points is vanishing, see (2.94), and the Wightman functions coincide, $G^+_\beta(t; y, y') = G^-_\beta(t; y, y')$. In Minkowski space-time this equality holds when $|t| < d(y, y')$ where $d(y, y')$ is the spatial distance between $y$ and $y'$.

These arguments show that there is a single function $\tilde{G}_\beta(z, y, y')$ defined by (7.140), (7.141) in the strip $-\beta < \Im z < \beta$, $-\infty < t < \infty$, except domains where the Wightman functions have singularities. In the Minkowski space-time $\tilde{G}_\beta(z, y, y')$ is analytic everywhere in the strip except two cuts at $\Im z = 0$, $|\Re z| \geq d(y, y')$.

The proof of the second property, Eq. (7.142), is based on (7.140), (7.141) and the identity

$$G^+_\beta(z - i\beta, y, y') = G^-_\beta(z, y, y'), \tag{12.205}$$

which follows from (12.201),(12.202).

**7.14** Representation (7.143) follows from (7.140), (7.141). The second property (ii) follows from (12.205). Equation (7.144) can be checked by using (7.143).

One should note that the field operators obey the equations $P_E\hat{\varphi}_E(x) = 0$ which results from the Lorentzian equation $P\hat{\varphi}(x) = 0$ under the Wick rotation. Therefore, $P_E(\partial^x)G(x, x') = 0$ if $x \neq x'$.

The time derivative of the step functions $\theta(\pm\tau)$ yields a delta-function and the corresponding terms are combined in the commutator of the operators on a constant-time hypersurface, see (2.18) and (12.34). The commutator produces the required delta-function in the r.h.s. of (7.144).

**7.15** Since a free field is system of harmonic oscillators it is enough to prove that for a single oscillator

$$\langle \hat{a}^+ \rangle_\beta = 0. \tag{12.206}$$

Consider to this aim a theory described the Hamiltonian

$$\hat{H}(j) = \omega \hat{a}^+ \hat{a} + j\hat{a}^+ + j^* \hat{a} \tag{12.207}$$

where an interaction with an external classical source is introduced. Define the partition function

$$Z(\beta, j) = \mathrm{Tr}\, e^{-\beta \hat{H}(j)}. \tag{12.208}$$

The average (12.206) can be written as

$$\langle \hat{a}^+ \rangle_\beta = -\beta^{-1} \frac{\partial}{\partial j} \ln Z(\beta, j)_{j=0}. \tag{12.209}$$

To compute $Z(\beta, j)$ we make the transformation

$$\hat{b}^+ = \hat{a}^+ + \omega^{-1} j^*, \qquad \hat{b} = \hat{a} + \omega^{-1} j, \tag{12.210}$$

where $\hat{b}^+$ and $\hat{b}$ obey the standard commutation relation, and get

$$\hat{H}(j) = \omega \hat{b}^+ \hat{b} - \frac{1}{\omega} |j|^2. \tag{12.211}$$

Therefore,

$$Z(\beta, j) = e^{\beta \omega^{-1} |j|^2} Z(\beta, j = 0). \tag{12.212}$$

Equation (12.206) follows from (12.212).

**7.16** We note that the extremum of $U(\varphi)$ at $\varphi = 0$ is unstable point. Stable points are at $\varphi = \pm\mu$. One can study the excitations near, say $\varphi = \mu$, and see that they behave as a free field with a mass $m^2 = 2\lambda\mu^2$.

The normalization conditions (7.146) identify the parameter $\mu$ with the average value of the field in quantum theory at the minimum of the effective potential. They also require that the mass of field excitations near the minimum coincides with the mass of the classical theory.

The corresponding CW-potential is given by the expression which follows from (7.90)

$$\Omega(\varphi) = \frac{a}{2}\varphi^2 + \frac{b}{12}\varphi^4 + \frac{\lambda^2}{64\pi^2}(3\varphi^2 - \mu^2)^2 \ln(3\varphi^2 - \mu^2). \tag{12.213}$$

For the classical part we use (7.86). For simplicity we put $\varrho = 1$, one can do this because redefinition of $\varrho$ is equivalent to change of constants $a$ and $b$. Conditions (7.146) yield

$$a = -\lambda\mu^2 + \frac{3\lambda^2\mu^2}{32\pi^2}(2\ln(2\mu^2) + 7), \tag{12.214}$$

$$b = 3\lambda - \frac{27\lambda^2}{32\pi^2}(2\ln(2\mu^2)+3). \tag{12.215}$$

After using (12.214), (12.215) in (12.213) we come to expression

$$\Omega(\varphi) = -\frac{1}{2}\mu^2\lambda\left(1 - \frac{21}{32\pi^2}\lambda\right)\varphi^2 + \frac{1}{4}\lambda\left(1 - \frac{27}{32\pi^2}\lambda\right)\varphi^4 + \varphi^4 + \frac{\lambda^2}{64\pi^2}(3\varphi^2-\mu^2)^2\ln\frac{3\varphi^2-\mu^2}{2\mu^2}. \tag{12.216}$$

Note that the coupling $\lambda$ is dimensionless so one may say that approximation we use is good enough when $\lambda \ll 1$. Thus, although quantum corrections in this model do affect the shape of the potential they do not change its global characteristics in the range of validity of this approximation. The potential still has two minima as in the classical theory.

**7.17** If $e^2 \gg \lambda$ quantum fluctuations of the gauge field dominate and quantum effects of the scalar field $\varphi$ can be neglected. The contribution of the vector field to the Coleman-Weinberg potential can be easily found.

When $|\varphi| = \phi$ the vector field has the mass $m^2 = m^2(\phi) = e^2\phi^2$. Quantization of a massive vector field was discussed in Exercises 2.5, 2.9. It follows from results of Exercise 2.9 that in Minkowski space-time the massive vector field is equivalent to three scalar bosons with the same mass, see Eq. (12.57) for the vacuum energy. Thus, for the Coleman-Weinberg potential one gets

$$\Omega(\phi) = \frac{a}{2}\phi^2 + \frac{b}{12}\phi^4 + 3\frac{e^2}{64\pi^2}\phi^2\ln(e^2\phi^2), \tag{12.217}$$

where constants $a$ and $b$ can be fixed as earlier, by conditions (7.146). This yields

$$\Omega(\phi) = -\frac{\lambda}{2}\mu^2\left(1 - \frac{3e^2}{16\pi^2\lambda}\right)\phi^2 + \frac{\lambda}{4}\left(1 - \frac{9e^4}{32\pi^2\lambda}\right)\phi^4 + \frac{3e^2}{64\pi^2}\phi^2\ln\frac{\phi^2}{\mu^2}. \tag{12.218}$$

The important feature of the Higgs model which makes it different from the pure scalar model is that quantum corrections can change properties of the effective potential already in the vacuum state. It is easy to see that at $e^2 > 16\pi^2\lambda/3$ the potential has a new minimum at $\phi = 0$ where the symmetry is not broken.

**7.18** The reason why the imaginary part of (7.148) can be connected with the decay probability is the following. Each complex frequency mode can be considered as a harmonic oscillator with a negative potential $-|\omega|^2 x^2/2$. Classically the particle cannot stay at the point $x = 0$ for a long time because of fluctuations. This means that states with complex energy modes cannot be stable. A time interval during which a quantum particle with a complex frequency $\omega$ stays at the point $x = 0$ is $\tau \sim |\omega|^{-1}$ (this follows from the fact that the wave-function of a particle changes with time as $e^{-i\omega t}$). The quantity $2\tau^{-1}$ can be considered as a probability for a given state to decay per unite time. Indeed, the probability to go for a unit time from a state $|\psi\rangle$ to itself is $p = |\langle\psi|e^{-i\omega}|\psi\rangle|^2 \simeq 1 - 2\omega$. The probability to go to a different state is $1 - p = 2\omega$.

For a large number of particles with imaginary energies $\omega_i$ being in a large volume $V$ one can define the probability for the entire system to decay in a unit volume per unit time as $\Gamma = 2V^{-1}\sum_{\Re\,\omega_i=0}|\omega_i|$. Therefore, $\Gamma = -2\Im\,\Omega(\phi)$. The sign by the imaginary part of the complex effective potential should be chosen so that to get the correct decay factor $\exp(-\Gamma)$ for probability per unit time.

To give a proof of (7.149) compute first the real part of $\Omega(\phi)$. One uses a zeta-function to regulate the series over the real frequencies

$$\zeta(s) \equiv \sum_{\Im\,\omega_i=0} \omega_i^{-2s} = \sum_{p_i^2>m^2} (p_i^2 - m^2)^{-s}, \tag{12.219}$$

where $p_i$ are the components of a momentum. By assuming that the system is in a box of a large volume one gets

$$\zeta(s) = \frac{4\pi V}{(2\pi)^3}\int_m^\infty dp\, p^2(p^2 - m^2)^{-s} = \frac{V}{(2\pi)^2}m^{3-2s}\frac{\Gamma(1-s)\Gamma(s-\frac{3}{2})}{\Gamma(-\frac{1}{2})}. \tag{12.220}$$

Therefore,

$$\frac{1}{2}\sum_{\mathrm{Im}\,\omega_i=0}\omega_i = \frac{1}{2}\lim_{s\to-\frac{1}{2}}\left(\zeta(s) - \frac{2\,\mathrm{Res}\,\zeta(-1/2)}{2s+1}\right) = \frac{V}{64\pi^2}m^4\ln m^2, \tag{12.221}$$

where $m^2 = -U''(\phi)$, and

$$\Re\,\Omega(\phi) = U(\phi) + \frac{1}{64\pi^2}(U''(\phi))^2\ln|U''(\phi)|. \tag{12.222}$$

It is instructive to compare this result with the corresponding expression (7.90) which we obtained earlier for $U''(\phi) > 0$. The imaginary part of the effective potential can be determined in terms of the following function:

$$\tilde{\zeta}(s) = \sum_{\Re\,\omega_i=0}(\omega_i^2)^{-s} = \sum_{p_i^2<m^2}(m^2 - p_i^2)^{-s}. \tag{12.223}$$

There is no problem with convergence of this sum at large $p^2$ and the direct computation yields

$$\tilde{\zeta}(s) = \frac{V}{(2\pi)^2}m^{3-2s}\frac{\Gamma(1-s)\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2}-s)}. \tag{12.224}$$

Thus,

$$\Im\,\Omega(\phi) = -\frac{1}{2V}\tilde{\zeta}(-1/2) = -\frac{\pi}{64\pi^2}(U''(\phi))^2. \tag{12.225}$$

By combining (12.222), (12.225) in (7.148) one arrives at (7.149). The advantage of (7.148) is that the effective potential can be an analytic function of $\phi$ at least in some part of the complex plane provided rules how to bypass the branching points in $\omega_i(\phi) = 0$ are formulated. Our convention, $\Im\,\omega_i < 0$, is achieved if $U''(\phi)$ is replaced to $U''(\varphi) - i\varepsilon$ where $\varepsilon > 0$ is a small addition.

**7.19** We give the proof of (7.150) for a Bose field (e.g., a charged scalar field) on a closed manifold. Generalization to the case of Fermions as well as to spaces with boundaries is straightforward. The effective action is given by the Ray-Singer formula (7.43). With the help of (12.190) the zeta-function (7.48) can be represented as

$$\zeta(s;P_E)=\frac{\varrho^{-2s}}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\sum_{l=-\infty}^{\infty}e^{-\sigma_l^2 t}\,\mathrm{Tr}\, e^{-tL(i\sigma_l)}. \tag{12.226}$$

At large temperatures (small $\beta$) the main contribution to terms with $\sigma_l\neq 0$ comes from the integration over small $t$. In this region one can use asymptotic expansion (6.18) for the heat kernel of $L(i\sigma_l)$.

By using (6.18) in (12.226) one can integrate over $t$, then one can perform the summation over $\sigma_l\neq 0$ by taking into account (6.19). This yields

$$\begin{aligned}\zeta(s;P_E)&\simeq\zeta(s;L(0))\\&+\frac{2\varrho^{-2s}\kappa^{n-1-2s}}{\Gamma(s)}\sum_{p=0}^{\infty}\zeta(2s+2p+1-n)\Gamma\left(s+p-\frac{n-1}{2}\right)a_{2p,s}\kappa^{-2p},\end{aligned} \tag{12.227}$$

$$a_{2p,s}=\sum_{m=p}^{2p}\frac{\Gamma(s+m-\frac{n-1}{2})}{\Gamma(s+p-\frac{n-1}{2})}(-1)^{m-p}a_{2(m-p),m}. \tag{12.228}$$

Here $\kappa=2\pi/\beta$, and $\zeta(z)$ is the Riemann $\zeta$-function. The quantities $a_{2m,p}$ are determined in (6.19). It follows from (6.20) and (12.228) that

$$a_{2p,s=0}=a_{2p},$$

where $a_{2p}$ are the coefficients of pseudo-trace expansion (6.17) on a closed $(n-1)$-dimensional space. Therefore, from (7.43) and (12.227) one finds

$$\begin{aligned}W[\phi_E]&\simeq-\zeta'(0;L(0))\\&\quad+2\kappa^{n-1}\sum_{p=0}^{n-1}\zeta(2p+1-n)\Gamma\left(p-\frac{n-1}{2}\right)a_{2p}\kappa^{-2p}+\frac{\beta}{\sqrt{\pi}}a_n\ln(\mu\beta)\\&=-\zeta'(0;L(0))-\frac{2^n}{\sqrt{\pi}\beta^{n-1}}\sum_{p=0}^{n-1}\zeta(n-2p)\Gamma\left(\frac{n-2p}{2}\right)a_{2p}\left(\frac{\beta}{2}\right)^{2p}\\&\quad+\frac{\beta}{\sqrt{\pi}}a_n\ln(\mu\beta).\end{aligned} \tag{12.229}$$

This equation coincides with (7.150) for the case of a Bose field. We omitted here a singular part which comes from the term with $2p=n$. This singularity is of an infrared origin and it appears because we do not take into account the presence of the mass gap (a smallest eigenvalue $\omega_0=\mu>0$ of the single-particle spectrum). If this is done carefully, the singularity does not appear.

**7.20** We use (12.229) to get for the model considered

$$W[\phi] \simeq \beta V\left(-\frac{\pi^2}{90}\beta^{-4} + \frac{1}{24}m^2\beta^{-2} + \frac{1}{32\pi^2}(m^2)^2 \ln(\mu\beta)\right). \quad (12.230)$$

An overall factor of $1/2$ is introduced here because the field is supposed to be real. The leading term in (12.230) determines the Stefan-Boltzmann law (7.134) for the gas of scalar particles. One can replace the mass gap $\mu$ to the mass of the field $m$.

This results in the following expression for the effective potential at high temperatures

$$\Omega(\phi, \beta) \simeq U(\varphi) + \frac{1}{24}m^2\beta^{-2} + \frac{1}{64\pi^2}(m^2)^2 \ln m^2\beta^2, \quad (12.231)$$

where $m^2 = U''(\phi)$. (Here the constant term has been omitted.) The main effect comes from the term quadratic in temperature. For model (7.145) one gets

$$\Omega(\phi, \beta) \simeq \frac{\lambda}{4}(\phi^2 - \mu^2)^2 + \frac{1}{24}\beta^{-2}\lambda(3\phi^2 - \mu^2). \quad (12.232)$$

As a consequence, if $T^2 > 4\mu^2$ the effective potential has a global minimum at $\phi = 0$.

**7.21** First, let us study the single-particle spectrum for electrons and positrons in a static electro-magnetic field. Consider the Dirac equation in Minkowski space-time

$$\left[\gamma^\mu(\partial_\mu - ieA_\mu) + m\right]\psi = 0, \quad (12.233)$$

where $A_\mu$ is a static vector-potential of the external electro-magnetic field, and $m$ is the electron mass. After the substitution $\psi_\omega(t, x) = e^{-i\omega t}\psi_\omega(x)$ the Dirac equation (12.233) is reduced to the following problem:

$$(\omega - H(\omega))\psi_\omega = 0, \quad (12.234)$$

$$H(\omega) = i\gamma_0(\gamma^k(\partial_k - ieA_k) + m) - eA_0. \quad (12.235)$$

The operator $H(\omega)$ is Hermitean. If $\lambda(\omega)$ are eigenvalues of $H(\omega)$, one has the algebraic problem

$$\omega - \lambda(\omega) = 0. \quad (12.236)$$

By using the chirality matrix $\gamma_\star$ let us bring (12.234) to a NLSP of form (6.3)

$$-(\gamma_\star i\gamma_0(\omega - H(\omega)))^2\phi_\omega = (\omega^2 - L(\omega))\phi_\omega = 0, \quad (12.237)$$

$$L(\omega) = -\mathscr{D}_k\mathscr{D}^k + m^2 - e^2A_0^2 + \frac{e}{2}(\mathscr{F} - 2i\gamma_0\gamma^k\partial_k A_0) - 2eA_0\omega, \quad (12.238)$$

where $\mathscr{D}_k = \partial_k - ieA_k$ and $\mathscr{F} = (\partial_j A_k - \partial_k A_j)\gamma^j\gamma^k$. Another way to get (12.237) is to start from $(\gamma^\mu(\nabla_\mu - ieA_\mu) - m)(\gamma^\nu(\nabla_\nu - ieA_\nu) + m)\phi = 0$. Spectrum pf (12.237) coincides with the spectrum of (12.234).

The effective action of the system at high temperatures can be written with the help of (7.150)

$$W[A] \sim -\frac{7\pi^{7/2}}{90\beta^3}a_0 - \frac{\pi^{3/2}}{6\beta}a_2 + \frac{a_4}{2\sqrt{\pi}}\beta\ln(m\beta). \tag{12.239}$$

The coefficients $a_k$ correspond to operator (12.238). By using formulae (6.18), (6.19), (6.20) and results of Chap. 5, one can find

$$(4\pi)^{3/2}a_0 = 4V, \tag{12.240}$$

$$(4\pi)^{3/2}a_2 = 4\int d^3x(-m^2 + 2e^2A_0^2), \tag{12.241}$$

$$(4\pi)^{3/2}a_4 = \frac{2e^2}{3}\int d^3x\, F_{\mu\nu}F^{\mu\nu}, \tag{12.242}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell stress tensor.

The total effective action for the static gauge field is (see (7.98))

$$\Gamma[A] = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} + W[A]. \tag{12.243}$$

By and taking into account (12.239)–(12.242) one gets

$$\Gamma[A] = -\beta\int d^3x\left(-\frac{c(T)}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2(T)A_0^2\right), \tag{12.244}$$

$$c(T) = 1 - \frac{e^2}{24\pi^2}\ln(T/\rho), \qquad M^2(T) = \frac{1}{3}e^2T^2. \tag{12.245}$$

It is assumed that (12.244) is given in terms of the renormalized fields and the charge (see discussion in Sect. 7.7). The functional (12.244) describes an effective three-dimensional gauge theory, where the component $A_0$ acquires an effective mass $M(T)$. As a result the Coulomb field of the charge in the plasma is screened at the distances larger than $1/M(T)$.

To be able to neglect the pair creation by the electric field we had to suppose that the mass $m$ is large. However, the mass gives rather non-interesting contributions to the effective action, and it has been dropped from the final answer.

**7.22** According to (7.151) one can give the following definition:

$$W_{(2k)}^{\mu_1,\mu_2,\ldots,\mu_{2k}}(x_1,x_2,\ldots,x_{2k}) = (2k)!\left.\frac{\delta^{2k}W[A]}{\delta A_{\mu_1}(x_1)\ldots\delta A_{\mu_{2k}}(x_{2k})}\right|_{A=0}. \tag{12.246}$$

The odd variations vanish at $A = 0$ as a result of the symmetry $W[A] = W[-A]$. Let us define the Green's function for the spinor field

$$(\not{D}(A) + m)G(A) = I \tag{12.247}$$

where we used symbolic notation $I$ for the delta-function. Variation of the Green's function is

$$\delta_A G(A) = (-ie)G(A)\gamma^\mu G(A)\delta A_\mu. \tag{12.248}$$

Provided a cutoff parameter is introduced one has from (7.91), (12.248)

$$\begin{aligned}\frac{\delta W[A]}{\delta A_\mu} &= -ie\,\mathrm{Tr}(G(A)\gamma^\mu),\\ \frac{\delta^2 W[A]}{\delta A_\mu \delta A_\nu} &= (-ie)^2\,\mathrm{Tr}(G(A)\gamma^\mu G(A)\gamma^\nu).\end{aligned} \tag{12.249}$$

Relations (12.246) are obtained by considering higher order derivatives at $A = 0$.

## 12.8 Chapter 8. Quantum Anomalies

**8.1** One should use the following properties of the $\gamma$-matrices in two dimensions:

$$\gamma_* = \frac{i}{2}\varepsilon_{\mu\nu}\gamma^\mu\gamma^\nu = i\gamma^1\gamma^2, \tag{12.250}$$

$$\gamma_*\gamma^\rho = -i\varepsilon^{\rho\nu}\gamma_\nu, \qquad [\gamma_\mu, \gamma_\nu] = -2i\varepsilon_{\mu\nu}\gamma_*. \tag{12.251}$$

This allows one to rewrite the Dirac operator $\not{D}(V, A)$, see (8.17), as

$$\not{D}(V, A) = i\gamma^\mu\left(\partial_\mu + V_\mu + \varepsilon_{\mu\rho}A^\rho\right) \equiv \not{D}(\bar{V}), \tag{12.252}$$

where we introduced an operator and a vector field

$$\not{D}(\bar{V}) = i\gamma^\mu(\partial_\mu + \bar{V}_\mu), \qquad \bar{V}_\mu = V_\mu + \varepsilon_{\mu\nu}A^\nu.$$

The chiral anomaly is easily calculated with the help of (8.26),

$$\delta_\lambda W[B] = -2i\zeta(0, \not{D}^2(\bar{V}), \gamma_*\lambda) = \frac{1}{2\pi}\int_{\mathscr{M}} d^2x\,\mathrm{tr}(\lambda F), \tag{12.253}$$

$$F = \varepsilon^{\mu\nu}F_{\mu\nu}, \qquad F_{\mu\nu} = [D_\mu, D_\nu], \qquad D_\mu = \partial_\mu + \bar{V}_\mu.$$

The trace refers to the gauge group indices only. Here we have ignored a possible contribution of the zero modes.

**8.2** Transformations (8.115) change the phase of the chiral effective action (8.114). The variations can be found by using formula (5.96),

$$\delta_\lambda W[B] = -i\delta\Phi(D_+(B)) = i\zeta(0, \not{D}^2(B), \gamma_*\lambda). \tag{12.254}$$

The corresponding computations were done in Exercise 5.9. For the theory in two dimensions which we use as an example one gets

$$\delta_\lambda W[B] = -\frac{1}{4\pi}\int_{\mathscr{M}} d^2x\,\mathrm{tr}(\lambda F), \tag{12.255}$$

where $F = \varepsilon^{\mu\nu}F_{\mu\nu}$, $F_{\mu\nu} = [D_\mu(B), D_\nu(B)]$, $F^+ = -F$. From (12.255) one gets the anomalous Noether condition for the current (8.116)

$$\partial_\mu\langle J^\mu\rangle + [\langle J^\mu\rangle, B_\mu] = -\frac{i}{4\pi}\varepsilon^{\mu\nu}F_{\mu\nu}. \tag{12.256}$$

A contribution of zero modes is ignored.

**8.3** The proof of the relation $T_{\mu\nu} = g_{\mu\rho} e^{\rho}_{a} T^{a}{}_{\nu}$, where $T_{\mu\nu}$ and $T^{a}{}_{\nu}$ are defined in (8.117), is based on definition of veilbeins (1.43). This implies that

$$\frac{\delta g_{\alpha\beta}(x)}{\delta e^{\sigma}_{a}(y)} = -\delta^{(n)}(x-y)(g_{\alpha\sigma} e^{a}_{\beta} + g_{\beta\sigma} e^{a}_{\alpha}).$$

**8.4** The relation $(D_+)^\dagger = D_-$ for operators (8.42) follows from the fact that $D_+$ can be written as

$$D_+\varphi = i\sqrt{2}\left(\nabla_\mu - \frac{i}{2} v_\mu\right)\bar{e}^\mu \varphi,$$

where $\nabla_\mu$ is a covariant derivative acting on vector fields. To prove this property one has to use definition of the complex basis (8.33) and relations

$$\bar{e}^\mu e_\nu + \bar{e}_\nu e^\mu = \delta^\mu_\nu, \qquad \nabla \bar{e} = i(\bar{e} \cdot v),$$

which follow from the definition.

**8.5** The prof of (8.118), (8.119) and the solution to the exercise follows straightforwardly from the definitions (1.4) and (1.10).

**8.6** The scalar Laplacian $L = -\nabla^2$ has a single normalized zero mode $\varphi_0$ on a compact closed manifold $\mathscr{M}$

$$\varphi_0 = \frac{1}{\sqrt{V}}, \tag{12.257}$$

where $V = \mathrm{vol}\,\mathscr{M}$ and $(\varphi_0, \varphi_0) = 1$. For anomaly calculation one uses Eq. (5.71). One has

$$\mathrm{Pr}^{(\alpha)}_n(\sigma) = (\varphi^{(\alpha)}_0, \sigma \varphi^{(\alpha)}_0), \tag{12.258}$$

where $\varphi^{(\alpha)}_0$ is the zero mode of the operator $L_\alpha$, see Eq. (8.75) for $n = 2$. From (12.257) one finds

$$\mathrm{Pr}^{(\alpha)}_n(\sigma) = \frac{1}{V_\alpha}\int \sqrt{g}\, d^2x\, \sigma(x) e^{-2\alpha\sigma(x)} = -\frac{1}{2}\frac{d}{d\alpha}\ln V_\alpha, \tag{12.259}$$

where $V_\alpha$ is the volume of the space with the metric $(g_\alpha)_{\mu\nu}$. Then instead of (8.97) the effective action takes the following form:

$$W[g] = W[\bar{g}] - \frac{1}{24\pi}\int_{\mathscr{M}} d^2x \sqrt{g}(R\sigma - (\nabla\sigma)^2) + \frac{1}{2}\ln\frac{V}{\bar{V}}. \tag{12.260}$$

**8.7** Consider a conformal transformation of $\mathscr{M}$ on a flat space $\bar{\mathscr{M}}$ with metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. Then the relation between the curvature $R$ and $\sigma$ takes the form,

$$R = -2\nabla^2\sigma = -2e^{-2\sigma}\partial^2\sigma, \tag{12.261}$$

were $\partial$ are usual derivatives. In the new frame effective action (8.97) on a space without boundaries takes the form

$$W[g] = -\frac{1}{24\pi}\int_{\bar{\mathscr{M}}} d^2x\big((\partial\sigma)^2 + \lambda e^{2\sigma}\big). \tag{12.262}$$

Variation over $\sigma$ yields the Liouville equation

$$\partial^2\sigma = \lambda e^{-2\sigma}. \tag{12.263}$$

The equations of the Liouville gravity are obtained by varying the metric and the field $\phi$

$$R = -\gamma\,\Delta\phi, \tag{12.264}$$

$$\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\frac{1}{2}(\nabla\phi)^2 - \frac{2}{\gamma}(g_{\mu\nu}\Delta\phi - \nabla_\mu\nabla_\nu\phi) + \frac{\mu}{2\gamma^2}g_{\mu\nu} = 0. \tag{12.265}$$

The trace of Eqs. (12.264) is

$$\Delta\phi = \frac{\mu}{2\gamma}. \tag{12.266}$$

After identifications (8.100) the last equation coincides with Liouville equation (12.263). By comparing (12.266) with (12.264) one concludes that solutions to the Liouville theory are constant curvature spaces.

**8.8** Formula (8.120) follows from the definition of the spin connection $w^{[s]}_\mu$, see Eqs. (1.51), (1.57).

**8.9** By acting in analogy with the case of scalar fields (see (8.75), (8.76)) one defines the family of operators

$$L_\alpha[g] = L[g_\alpha] = (\not{D}[g_\alpha])^2 = \big(e^{\frac{n+1}{2}\alpha\sigma}\not{D}[g]e^{-\frac{n-1}{2}\alpha\sigma}\big)^2, \tag{12.267}$$

$$\frac{d}{d\alpha}\operatorname{Tr} e^{-tL_\alpha} = -2t\operatorname{Tr}\big(\sigma e^{-tL_\alpha}\big). \tag{12.268}$$

Equation (12.268) coincides with (8.76). Therefore, from (5.59) and (5.70) one finds

$$\begin{aligned} W[\bar{g}] - W[g] &= -\big(\ln\det\not{D}[\bar{g}] - \ln\det\not{D}[g]\big) = -\frac{1}{2}\big(\zeta'(0, L[\bar{g}]) - \zeta'(0, L[g])\big) \\ &= \int_0^1 d\alpha\,\zeta\big(0, (\not{D}[g_\alpha])^2, \sigma\big). \end{aligned} \tag{12.269}$$

If one ignores contribution of zero modes the spinor action differs from the effective action of a massless scalar field only by the sign ($\zeta(0, L_\alpha, \sigma)$ is given by (8.80) with a reversed sign on the right hand side).

**8.10** This is a straightforward calculation based on (8.19).

## 12.9 Chapter 9. Vacuum Energy

**9.1** For the model on a circle the single particle energies are $\omega_k = (2\pi/l)k$, $k = 1, 2, \ldots$. The degeneracy is $d_k = 2$.

**First method.** Sum (9.94) can be easily computed

$$E_0(\epsilon, l) = -\partial_\epsilon \left( \frac{1}{2} \sum_k d_k \, e^{-\epsilon \omega_k} \right) = -\partial_\epsilon \frac{1}{e^{\epsilon a} - 1}, \tag{12.270}$$

where $a = 2\pi/l$. By using formula

$$\frac{1}{e^z - 1} \simeq \frac{1}{z} - \frac{1}{2} + \frac{z}{12}$$

one finds

$$E_0(\epsilon, l) = \frac{l}{2\pi\epsilon^2} - \frac{\pi}{6l} + O(\epsilon^2). \tag{12.271}$$

To understand the physical meaning of this result consider the energy density

$$\rho(\epsilon, l) = E_0(\epsilon, l)/l = \frac{1}{2\pi\epsilon^2} - \frac{\pi}{6l^2} + O(\epsilon^2). \tag{12.272}$$

It has yet a divergent part, but as one can note the divergence does not depend on the size of the circle $l$. This means that the difference of energy densities on the circles of different sizes is a well-defined quantity. In particular, in the limit of infinite $l$ (12.272) represents the energy on the line

$$\rho_0(\epsilon) = \frac{1}{2\pi\epsilon^2}. \tag{12.273}$$

Therefore, the difference

$$\rho_{\mathrm{subtr}}(l) = \lim_{\epsilon \to 0} (\rho(\epsilon, l) - \rho_0(\epsilon)) = -\frac{\pi}{6l^2} \tag{12.274}$$

remains finite in the limit $\epsilon \to 0$.

The energy and the force which appear under the change of the size of the circle are, respectively,

$$E_0(l) = -\frac{\pi}{6l}, \qquad F(l) = -\partial E_0(l)/\partial l = -\frac{\pi}{6l^2}. \tag{12.275}$$

**Second method.** By using explicit expression (2.99) for the Wightman function and the point-splitting method, see Sect. 2.7, one defines

$$\rho(x) = \langle T_{00}(x) \rangle = \lim_{x' \to x} \frac{1}{2} [\partial_t \partial_t' + \partial_{\mathbf{x}} \partial_{\mathbf{x}}'] G^+(x, x'). \tag{12.276}$$

One gets

$$\langle T_{00}(x) \rangle = -\frac{\pi}{2l^2} \frac{1}{\sin^2 \frac{\pi(\mathbf{x}-\mathbf{x}')}{l}}. \tag{12.277}$$

This expression corresponds to the points with coordinates $x = (t, \mathbf{x})$, $x' = (t, \mathbf{x}')$. In the limit when $\mathbf{x}'$ approaches $\mathbf{x}$ we find

$$\langle T_{00}(x)\rangle = -\frac{1}{2\pi(\mathbf{x}-\mathbf{x}')^2} - \frac{\pi}{6l^2} + O\big((\mathbf{x}-\mathbf{x}')^2\big). \tag{12.278}$$

The first, divergent term in r.h.s. of this expression does not depend on the size $l$. This term is the same as the divergence in the stress energy tensor on the line. One can subtract from the energy density its value on the line and come to the finite result

$$\langle T_{00}^{\mathrm{subtr}}(x)\rangle = -\frac{\pi}{6l^2}, \tag{12.279}$$

which agrees with (12.274).

**9.2** The spectrum of single-particle energies on the Einstein universe is determined by the spectrum of the Laplace operator $\Delta$ on $S^3$. The eigen values $\lambda_n$ of $\Delta$ are $\lambda_n = r^{-2}n(n+2)$ where $n = 0, 1, 2, \dots$. Each eigenvalue is degenerate and has multiplicity $d_n = (n+1)^2$. The single-particle energies are determined by equation $\omega_n^2 = \lambda_n + m^2$. If $m = r^{-1}$ we find $\omega_n = (n+1)/r$. The corresponding vacuum energy can be computed by using formulas from Chap. 5. For a real field $E_0 = (240r)^{-1}$. This result can be compared with the vacuum energy of a massless field on a circle. This model can be considered as a two-dimensional analog of the Einstein universe, $R^1 \times S^1$, and its vacuum energy, as we saw was negative, see (12.275). Thus, the Casimir force on $S^1$ is the force of attraction, while the force on $S^3$ is repulsive.

**9.3** The spectrum of single-particle energies of the Weyl spinors is $\omega_n = n + 3/2$ with the degeneracy $(n+1)(n+2)$. The eigenvalues for $\psi$ and $\psi^c$ coincide because the gauge fields are absent. By formula (9.1) one gets $E_0 = r^{-1}(\zeta(-3, 3/2) - \frac{1}{4}\zeta(-1, 3/2)) = -17/(960r)$.

**9.4** Any two anticommuting Majorana spinors, $\psi$ and $\xi$, obey the identity

$$\bar{\psi}\xi = \bar{\xi}\psi, \qquad \bar{\psi}\gamma^\mu\xi = -\bar{\xi}\gamma^\mu\psi. \tag{12.280}$$

By using this and neglecting all surface terms one can calculate the SUSY variations of four terms in action (9.45):

$$\begin{aligned}
\delta_1 &= \int d^2x\, \bar{\epsilon}\psi\partial_\mu^2\varphi, \\
\delta_2 &= \int d^2x\, U'(\bar{\epsilon}\gamma^\mu\partial_\mu\psi + \bar{\epsilon}\psi U), \\
\delta_3 &= \int d^2x(-\bar{\epsilon}\psi\partial_\mu^2\varphi + U\bar{\epsilon}\gamma^\mu\partial_\mu\psi), \\
\delta_4 &= \int d^2x(-UU'\bar{\epsilon}\psi).
\end{aligned} \tag{12.281}$$

Now it is obvious that $\delta_1 + \delta_2 + \delta_3 + \delta_4 = 0$ up to surface terms.

**9.5** The linearized SUSY transformations with the $\epsilon_+$ parameter read:

$$\delta\phi = i\psi_-\epsilon_+, \qquad \delta\psi_+ = (\partial_x\chi - U'(\phi)\chi)\epsilon_+, \qquad \delta\psi_- = \partial_0\chi\epsilon_+. \quad (12.282)$$

To prove that the boundary conditions, (9.77) or (9.78) are SUSY invariant, one has to show that the SUSY variations of the fields satisfy the same boundary conditions as the fields themselves. This statement is obvious for the $\psi_+$ and $\chi$ boundary conditions from set A, and for the boundary conditions on $\chi$ from set B. Since $\partial_0\chi$ satisfy the same boundary conditions as $\chi$, the boundary conditions on $\psi_-$ in both sets are also SUSY invariant. The only remaining boundary condition is the one on $\psi_+$ in set B. We have $(\partial_x + U'(\phi))\delta\psi_+ = D_-D_+\chi\epsilon_+$. This expression vanishes on the boundary due to (9.75) and Dirichlet boundary condition on $\chi$.

**9.6** The anti-kink solution satisfies the Bogomolny equation (9.61) with the plus sign (instead of minus for the kink). As a consequence, the anti-kink is invariant under the SUSY transformations with $\epsilon_+ = 0$. It is easy to check then that in the SUSY boundary conditions (9.77) and (9.78) roles of $\psi_+$ and $\psi_-$ are interchanged, and the sign in front of $U'$ is reversed. Formula (9.88) receives an overall minus sign. The anti-kink solution has the same mass shift (9.89) as the kink solution.

**9.7** Let us introduce a bosonic background field $\phi$ which now is not supposed to be static or to satisfy the equations of motion. One-loop quantum corrections are governed by the part of the classical action which is quadratic in quantum fluctuations $\chi$, $\psi$

$$\begin{aligned}[S + S^{\text{bou}}]_2 = &-\frac{1}{2}\int_{\mathcal{M}} d^2x\big[\chi(-\partial_\mu^2 + U'^2 + UU'')\chi + \bar{\psi}(\not{\partial} + U')\psi\big] \\ &+ \int_{\partial\mathcal{M}} dt\, n_x\left[\frac{1}{2}\chi(\partial_x - U')\chi - \frac{1}{4}\bar{\psi}\psi\right]. \end{aligned} \quad (12.283)$$

Here, the superpotential and its derivatives depend on the background field $\phi$. The boundary part vanishes under the conditions (9.77) or (9.78), and the bulk part defines an eigenvalue problem for certain hyperbolic operators. To make these operators elliptic, we perform a Wick rotation, $\partial_0 \to i\partial_2$. The spatial coordinate will be denoted as $x^1$. In the bosonic sector we have

$$L_b = -\partial_1^2 - \partial_2^2 + U'^2 + UU''. \quad (12.284)$$

In the fermionic sector we obtain a Dirac operator

$$\not{D}_E = \begin{pmatrix} \partial_1 + U' & -i\partial_2 \\ i\partial_2 & -\partial_1 + U' \end{pmatrix} \quad (12.285)$$

which is not Hermitian but is unitarily equivalent to its conjugate,

$$\not{D}_E^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \not{D}_E \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (12.286)$$

Due to this property, all complex eigenvalues of $\not{D}_E$ come in pairs with their conjugates, and one can write

$$\ln\det \not{D}_E = \frac{1}{2}\ln\det(\not{D}_E^{\dagger}\not{D}_E). \tag{12.287}$$

The operator

$$L_f = \not{D}_E^{\dagger}\not{D}_E = -\partial_1^2 - \partial_2^2 + \begin{pmatrix} -\partial_1 U' + U'^2 & i\partial_2 U' \\ -i\partial_2 U' & \partial_1 U' + U'^2 \end{pmatrix} \tag{12.288}$$

is suitable for the heat-kernel analysis of the one-loop divergences. In the zeta-function regularization the divergent part of the one-loop effective action reads

$$W_{\text{div}} = -\frac{1}{2s}\left(a_2(L_b) - \frac{1}{2}a_2(L_f)\right), \tag{12.289}$$

where an extra $1/2$ in front of $a_2(L_f)$ appeared since we are dealing with real Majorana spinors. Another complication appears because $\psi_+$ and $\psi_-$ satisfy different types of boundary conditions in both sets (9.77) and (9.78), and $\psi_\pm$ are mixed up in (12.288) due to the off-diagonal terms. Therefore, we have mixed boundary conditions (3.45) in the fermionic sector, which cannot be reduced to a sum of Dirichlet and Neumann problems if $\partial_2 U \neq 0$. We have not calculated the heat kernel expansion for mixed boundary conditions. At least for the leading coefficients this may be done by the same methods which we used in Sect. 4.5 above (see [49, 52]). First, let us extend $\mathcal{S}$ to the whole space of spinors by means of the equation $\Pi_N \mathcal{S} \Pi_N = \mathcal{S}$ (see (3.45) for definitions). This yields in our case

$$\mathcal{S}_f = \begin{pmatrix} 0 & 0 \\ 0 & -n_1 U' \end{pmatrix} \tag{12.290}$$

for the conditions (9.77) (when $\Pi_N$ is a projector on $\psi_-$), and

$$\mathcal{S}_f = \begin{pmatrix} n_1 U' & 0 \\ 0 & 0 \end{pmatrix} \tag{12.291}$$

for (9.78) (when $\Pi_N$ is a projector on $\psi_+$). Obviously, boundary conditions (3.45) are equivalent to (9.77) or (9.78) depending on the choice of $\Pi_N$.

We have, strictly speaking, a new type of boundary conditions for which the heat kernel expansion is not known. Therefore, we should extend our previous results for $a_2$ with Dirichlet and Neumann boundary conditions to spectral problems with mixed conditions (3.45). Let us count the boundary invariants which may appear in $a_2(L)$ for mixed boundary conditions. The extrinsic curvature of the boundary is zero, so there is just one quantity of the mass dimension one, and this is $\mathcal{S}$, as for Robin boundary conditions. It does not make sense to multiply $\mathcal{S}$ by $\Pi_D$ or $\Pi_N$ (both projectors have a unit mass dimension), so that the number of possible terms remains the same as for Robin boundary conditions. Namely, we have only the boundary integral of $\text{tr}(\mathcal{S})$. The coefficient in front is easily recovered by considering a particular case when the Dirichlet space shrinks to zero and comparing to (4.75) and (4.89). The result, which actually unifies (4.75) and (4.72) for flat boundaries, flat bulk, and unit smearing function, reads

$$a_2(L) = (4\pi)^{-n/2}\left[\int_{\mathscr{M}} d^n x\, \mathrm{tr}(E) + \int_{\partial\mathscr{M}} d^{n-1}x\, \mathrm{tr}(2\mathscr{S})\right]. \tag{12.292}$$

In the bosonic sector we have $\mathscr{S}_b = -n_1 U'$ and $\mathscr{S}_b = 0$ for (9.77) and (9.78) respectively. The matrix valued potentials $E$ (not to be confused with the vacuum energy) are easily extracted from (12.284) and (12.288). By collecting everything together, we obtain the same result for both sets of boundary conditions:

$$W_{\mathrm{div}} = -\frac{1}{2s}\frac{1}{4\pi}\left[\int_{\mathscr{M}} d^2x(-UU'') - \int_{\partial\mathscr{M}} dt\, n_1 U'\right]. \tag{12.293}$$

Both bulk and boundary divergences can be removed by renormalizing the superpotential

$$\delta U = \frac{1}{8\pi s}U'' \tag{12.294}$$

in classical bulk and boundary actions (9.45) and (9.90). In the particular case of $\varphi^4$ model, Eq. (9.68), this is just the mass renormalization.

## 12.10 Chapter 10. Open Strings and Born-Infeld Action

**10.1** Since we assumed that the target space metric $G_{\mu\nu}$ is constant, the bulk term is easy. It immediately yields the first line of (10.2). The boundary term gives in the second order in $\xi$

$$\int_{\partial\mathscr{M}} d\tau\left((\partial_\nu A_\mu(\bar{X}))\xi^\nu\partial_\tau\xi^\mu + \frac{1}{2}\partial_\tau\bar{X}^\mu\cdot\xi^\nu\xi^\rho\partial_\nu\partial_\rho A_\mu(\bar{X})\right). \tag{12.295}$$

Let us remind that $\partial_\mu$ denotes a partial derivative with respect to $\bar{X}^\mu$. Consider the first term in (12.295)

$$\begin{aligned}
&\int_{\partial\mathscr{M}} d\tau(\partial_\nu A_\mu)\xi^\nu\partial_\tau\xi^\mu \\
&\quad = \frac{1}{2}\int_{\partial\mathscr{M}} d\tau\big((\partial_\nu A_\mu)\xi^\nu\partial_\tau\xi^\mu - \xi^\mu\partial_\tau(\partial_\nu A_\mu\xi^\nu)\big) \\
&\quad = \frac{1}{2}\int_{\partial\mathscr{M}} d\tau\big((\partial_\nu A_\mu)\,\xi^\nu\partial_\tau\xi^\mu - \xi^\mu\partial_\tau\xi^\nu\cdot\partial_\nu A_\mu - \xi^\mu\xi^\nu\partial_\nu\partial_\rho A_\rho\cdot\partial_\tau\bar{X}^\mu\big) \\
&\quad = \frac{1}{2}\int_{\partial\mathscr{M}} d\tau\big(F_{\nu\mu}(\bar{X})\xi^\nu\partial_\tau\xi^\mu - \xi^\mu\xi^\nu\partial_\nu\partial_\rho A_\rho\cdot\partial_\tau\bar{X}^\mu\big),
\end{aligned} \tag{12.296}$$

where we used integration by parts (there is no boundary in the $\tau$-direction) and the chain rule $\partial_\tau A_\mu(\bar{X}) = \partial_\nu A_\mu(\bar{X})\cdot\partial_\tau\bar{X}^\nu$. The first term on the last line of (12.296) already coincides with corresponding term in (10.2). The second term combines with the second term in (12.295) to give the remaining term in the second-order boundary action (10.2).

**10.2** The statement of this exercise consists of two parts. First, one has to prove that

$$\Delta G(x,x') = \delta(x,x') \tag{12.297}$$

on $\mathscr{M}$. The second and the third terms in (10.23) do not have singularities on $\mathscr{M}$ and do not contribute to (12.297). The first term in (10.23) is the standard Green's function on $\mathbb{R}^2$ (cf. (2.68)) which generates the delta-function in (12.297). The boundary conditions (10.6) can be checked by a straightforward computation. We have

$$\partial_\sigma G|_{\sigma=0} = \frac{1}{(1+\Gamma^2)|z-z'|^2}(-4\sigma'\Gamma^2 - 4\Gamma(\tau-\tau')),$$
$$\partial_\tau G|_{\sigma=0} = \frac{1}{(1+\Gamma^2)|z-z'|^2}(4(\tau-\tau') + 4\Gamma\sigma').$$

These equations yield

$$(\partial_\sigma G + \Gamma\partial_\tau G)|_{\sigma=0} = 0,$$

which proves the boundary conditions (10.6) in the particular case we consider in this exercise.

**10.3** First we have to check that $\Delta_z G_S(z,z') = \delta(z-z')$ when both $z$ and $z'$ are inside the manifold. Let us apply $\Delta_z$ to the Dyson equation (10.24). The first term on the right hand side produces the delta function. The second term produces a contribution containing $\delta(z-\tau'')$, which is identically zero since $z$ is inside the manifold, and $\tau''$ is on the boundary.

The Green's function $G_0$ can be obtained from the Green's function (10.23) by simply taking $\Gamma = 0$,

$$G_0(z,z') = -\frac{1}{4\pi}\left[\ln|z-z'|^2 + \ln|z-\bar{z}'|^2\right]. \tag{12.298}$$

Let us differentiate this function with respect to $\sigma$ when the second argument in on the boundary.

$$\partial_\sigma G_0(z;\tau'',0) = -\frac{1}{\pi}\frac{\sigma}{(\tau-\tau'')^2+\sigma^2}. \tag{12.299}$$

Now let us differentiate Eq. (10.24) with respect to $\sigma$ and then put $\sigma = 0$. The first term on the right hand side gives zero due to the boundary condition on $G_0$. The second term contributes because of a singularity on the Green's function for coinciding arguments. The integral over $\tau''$ is performed by using the delta-function which appears due to the identity

$$\lim_{\sigma\to 0}\frac{\sigma}{(\tau-\tau'')^2+\sigma^2} = \pi\delta(\tau-\tau'').$$

This proves boundary condition (10.25).

Perturbation series which solve (10.24) can be obtained if one takes $G_S = G_0$ as a zeroth approximation and then iterates (10.24):

$$
\begin{aligned}
G_S(z, z')
&= G_0(z, z') + \int d\tau_1\, G_0(z; \tau_1, 0) S(\tau_1) G_0(\tau_1, 0; z') \\
&\quad + \int d\tau_1 \int d\tau_2\, G_0(z; \tau_1, 0) S(\tau_1) G_0(\tau_1, 0; \tau_2, 0) S(\tau_2) G_0(\tau_2, 0; z') + \cdots .
\end{aligned}
\tag{12.300}
$$

**10.4** We give the proof for the case when $\Gamma = 0$ and leave its extension to the case $\Gamma \neq 0$ to the reader. The correction to the action in the linear order in $\mathscr{S}$ can be computed as a variation of the determinant

$$W_1(\mathscr{S}) = \frac{1}{2}\delta_{\mathscr{S}} \ln\det\Delta = \frac{1}{2}\operatorname{Tr} G\delta_{\mathscr{S}}\Delta = -\frac{1}{2}\operatorname{Tr}\Delta\delta_{\mathscr{S}} G \tag{12.301}$$

where $G = G_0$ and

$$\delta_{\mathscr{S}} G(z, z') = \int d\tau_1\, G_0(z; \tau_1, 0)\mathscr{S}(\tau_1) G_0(\tau_1, 0; z')$$

see Eq. (12.300). These relations give (10.26).

**10.5** Since we restricted ourselves to a flat geometry, we expect that the divergence of the effective action is proportional to a surface integral of $\operatorname{tr}(b_2(\Gamma)\mathscr{S})$. Since an overall coefficient plays no role we need to reproduce in the divergent part the structure $(1 + \Gamma^2)^{-1}$.

The divergent term we are looking for does not contain derivatives of $\Gamma$ and we can impose the condition $\Gamma = const$, as in (10.23). Since we need the terms which are linear in $\mathscr{S}$ we can use a perturbation expansion in $\mathscr{S}$ of Exercise 10.4.

Note that we keep $\tau' - \tau$ fixed and use this difference as a regularization parameter (by assuming that $\tau > \tau'$). The singular part of the propagator is

$$G(\tau, 0; \tau', 0) = -\frac{1}{2\pi}\ln|\tau - \tau'|^2 [1 + \Gamma^2]^{-1}, \tag{12.302}$$

see (10.23). This yields the divergent part of the effective action to the linear order in $\mathscr{S}$,

$$W_1(\mathscr{S})_{\text{div}} \propto \ln|\tau - \tau'|^2 \int_{\partial\mathscr{M}} d\tau\, \operatorname{tr}(\mathscr{S}(\tau)[1 + \Gamma^2]^{-1}), \tag{12.303}$$

which correctly reproduces the functional dependence of $a_2(L)$ on $\mathscr{S}$ and $\Gamma$. A precise numerical coefficient can be restored by comparing this result to the case $\Gamma = 0$ (cf. (4.75), (4.89)).

**10.6** Suppressing the target space vector indices we write

$$
\begin{aligned}
\sqrt{\det(1 + iF)} &= \exp\frac{1}{2}\operatorname{tr}\ln(1 + iF) \\
&= \exp\frac{1}{4}\operatorname{tr}\ln[(1 + iF)(1 - iF)] = \exp\frac{1}{4}\operatorname{tr}\ln(1 + F^2),
\end{aligned}
$$

where we used that $F_{\mu\nu}$ is antisymmetric, so that only even powers $F$ contribute to the trace, and, consequently, reversing the sign in front of $F$ does not change the result. Next,

$$\delta I_{\rm BI} = \int d^N X \sqrt{\det(1+iF)}\frac{1}{4}\,{\rm tr}\left[\frac{\delta F^2}{1+F^2}\right].$$

Now it is obvious that the field equations following from the Born-Infeld action are equivalent to conditions (10.16) where $\beta_\mu^A$ is given by (10.15).

## 12.11 Chapter 11. Noncommutative Geometry and Field Theory

**11.1** It is enough to demonstrate the associativity on plane waves. By using (11.7), we have

$$(e^{ikx}\star e^{ipx})\star e^{iqx} = e^{i(k+p+q)x}e^{-\frac{i}{2}(k\wedge p+(k+p)\wedge q)},$$
$$e^{ikx}\star (e^{ipx}\star e^{iqx}) = e^{i(k+p+q)x}\,e^{-\frac{i}{2}(k\wedge (p+q)+p\wedge q)}.$$

The two lines above coincide. To prove (11.5), one has to integrate by parts and use the property $\theta^{\mu\nu}\partial_\mu\partial_\nu = 0$.

**11.2** To the order we are interested in, the heat trace reads

$$\begin{aligned} K(P,t) &= {\rm Tr}(e^{-tP}) \\ &\simeq {\rm Tr}\left(-\frac{\lambda t}{6}(R(\phi\star\phi)+L(\phi\star\phi)+L(\phi)R(\phi))e^{-t\Delta}\right)e^{-m^2t} \\ &= -\frac{\lambda t}{6}\,e^{-m^2t}\,(T(1,\phi\star\phi)+T(\phi\star\phi,1)+T(\phi,\phi)). \end{aligned}$$

By using (11.22) and (11.24) one transforms the last line in the equation above to

$$-\frac{\lambda}{12\pi}\int d^2x\,\phi\star\phi e^{-m^2t} - \frac{\lambda t\,(\det\theta)^{-1}}{6(2\pi)^2}e^{-m^2t}\left[\int d^2x\,\phi\right]^2.$$

The second non-local term in this formula is $\mathscr{O}(t)$, and, therefore, it does not generate any divergence. The first term is $\mathscr{O}(t^0)$ and contributes to a divergence. Due to (11.5), this divergence can be removed by a mass renormalization. It is easy to see that the divergent part is $2/3$ of that in the commutative $\varphi^4$ in two dimensions.

**11.3** The commutator $[L(f), i\gamma^a L(e_a^\mu)\partial_\mu]$ contains a first-order unbounded part $-i\gamma^a L(e_a^\mu\star f)\partial_\mu$ in contradiction to the axiom (c) of spectral triples. Calculations of the heat kernel expansion for such operators are very involved [72, 245]. This also explains some difficulties in construction of gravity theories on Moyal spaces.

# References


1. Abouelsaood, A., Callan, C.G., Nappi, C.R., Yost, S.A.: Open strings in background gauge fields. Nucl. Phys. B **280**, 599 (1987)
2. Adler, S.L.: Axial vector vertex in spinor electrodynamics. Phys. Rev. **177**, 2426–2438 (1969)
3. Adler, S.L.: Einstein gravity as a symmetry breaking effect in quantum field theory. Rev. Mod. Phys. **54**, 729 (1982)
4. Allen, B.: Phase transitions in de Sitter space. Nucl. Phys. B **226**, 228 (1983)
5. Alonso Izquierdo, A., et al.: Lectures on the mass of topological solitons. hep-th/0611180 (2007)
6. Alvarez-Gaume, L., Ginsparg, P.H.: The topological meaning of nonabelian anomalies. Nucl. Phys. B **243**, 449 (1984)
7. Alvarez-Gaume, L., Ginsparg, P.H.: The structure of gauge and gravitational anomalies. Ann. Phys. **161**, 423 (1985)
8. Alvarez-Gaume, L., Witten, E.: Gravitational anomalies. Nucl. Phys. B **234**, 269 (1984)
9. Alvarez-Gaume, L., Della Pietra, S., Moore, G.W.: Anomalies and odd dimensions. Ann. Phys. **163**, 288 (1985)
10. Ambjorn, J., Wolfram, S.: Properties of the vacuum. 1. Mechanical and thermodynamic. Ann. Phys. **147**, 1 (1983)
11. Andrianov, A.A., Bonora, L.: Finite-mode regularization of the fermion functional integral. Nucl. Phys. B **233**, 232 (1984)
12. Andrianov, A.A., Novozhilov, Y.: The chiral bosonization in nonabelian gauge theories. Phys. Lett. B **153**, 422 (1985)
13. Aref'eva, I.Y., Belov, D.M., Koshelev, A.S.: Two-loop diagrams in noncommutative $\phi^4_4$ theory. Phys. Lett. B **476**, 431–436 (2000). hep-th/9912075
14. Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian Geometry 1. Math. Proc. Camb. Philos. Soc. **77**, 43 (1975)
15. Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian Geometry 2. Math. Proc. Camb. Philos. Soc. **78**, 405 (1976)
16. Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian Geometry 3. Math. Proc. Camb. Philos. Soc. **79**, 71 (1980)
17. Avramidi, I.G.: Covariant techniques for computation of the heat kernel. Rev. Math. Phys. **11**, 947–980 (1999). hep-th/9704166
18. Avramidi, I.G.: Heat Kernel and Quantum Gravity. Springer, Berlin (2000)
19. Avramidi, I.G., Esposito, G.: Heat-kernel asymptotics with generalized boundary conditions. Class. Quantum Gravity **15**, 281–297 (1998). hep-th/9701018
20. Avramidi, I.G., Esposito, G.: Lack of strong ellipticity in Euclidean quantum gravity. Class. Quantum Gravity **15**, 1141–1152 (1998). hep-th/9708163


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Theoretical and Mathematical Physics,
DOI 10.1007/978-94-007-0205-9, 

21. Avramidi, I.G., Esposito, G.: Gauge theories on manifolds with boundary. Commun. Math. Phys. **200**, 495–543 (1999). hep-th/9710048
22. Balian, R., Bloch, C.: Distribution of eigenfrequencies for the wave equation in a finite domain. 1. Three-dimensional problem with smooth boundary surface. Ann. Phys. **60**, 401–447 (1970)
23. Balian, R., Duplantier, B.: Eelectromagnetic waves near perfect conductors. 1. Multiple scattering expansions. Distribution of modes. Ann. Phys. **104**, 300 (1977)
24. Bardeen, W.A., Zumino, B.: Consistent and covariant anomalies in gauge and gravitational theories. Nucl. Phys. B **244**, 421 (1984)
25. Barut, A., Raczka, R.: Theory of Group Representations and Applications. PWN, Warszawa (1977)
26. Barvinsky, A.O., Vilkovisky, G.A.: The generalized Schwinger-DeWitt technique in gauge theories and quantum gravity. Phys. Rep. **119**, 1–74 (1985)
27. Barvinsky, A.O., Vilkovisky, G.A.: Beyond the Schwinger-Dewitt technique: converting loops into trees and in-in currents. Nucl. Phys. B **282**, 163–188 (1987)
28. Barvinsky, A.O., Vilkovisky, G.A.: Covariant perturbation theory. 2: Second order in the curvature. General algorithms. Nucl. Phys. B **333**, 471–511 (1990)
29. Bastianelli, F., Corradini, O., Pisani, P.A.G., Schubert, C.: Scalar heat kernel with boundary in the worldline formalism. J. High Energy Phys. **10**, 095 (2008). 0809.0652
30. Batemen, H., Erdélyi, A.: Higher Transcendental Functions, vol. 1. McGraw-Hill, New York (1953)
31. Bell, J.S., Jackiw, R.: A PCAC puzzle: $\pi^0 \to \gamma\gamma$ in the sigma model. Nuovo Cimento A **60**, 47–61 (1969)
32. Belyaev, D.V., van Nieuwenhuizen, P.: Rigid supersymmetry with boundaries. J. High Energy Phys. **04**, 008 (2008). 0801.2377
33. Beneventano, C.G., Santangelo, E.M.: Spectral functions of the Dirac operator under local boundary conditions (2004). hep-th/0405221
34. Berline, N., Getzler, E., Vergne, M.: Heat Kernels and Dirac Operators. Springer, Berlin (2004)
35. Bertlmann, R.A.: Anomalies in Quantum Field Theory. International Series of Monographs on Physics, vol. 91, 566 pp. Clarendon, Oxford (1996)
36. Bertozzini, P., Conti, R., Lewkeeratiyutkul, W.: Non-commutative geometry, categories and quantum physics (2008). 0801.2826
37. Birrell, N.D., Davies, P.C.: Quantum Fields in Curved Space. Cambridge University Press, Cambridge (1982)
38. Blau, S., Visser, M., Wipf, A.: Zeta functions and the Casimir energy. Nucl. Phys. B **310**, 163 (1988)
39. Blau, S.K., Visser, M., Wipf, A.: Analytical results for the effective action. Int. J. Mod. Phys. A **6**, 5409–5433 (1991)
40. Bogolyubov, N.N., Shirkov, D.V.: Introduction to the Theory of Quantized Fields. Intersci. Monogr. Phys. Astron. (1959)
41. Bogomolny, E.B.: Stability of classical solutions. Sov. J. Nucl. Phys. **24**, 449 (1976)
42. Bordag, M.: Vacuum energy in smooth background fields. J. Phys. A **28**, 755–766 (1995)
43. Bordag, M., Kirsten, K., Dowker, J.S.: Heat kernels and functional determinants on the generalized cone. Commun. Math. Phys. **182**, 371–394 (1996). hep-th/9602089
44. Bordag, M., Kirsten, K., Vassilevich, D.: On the ground state energy for a penetrable sphere and for a dielectric ball. Phys. Rev. D **59**, 085011 (1999). hep-th/9811015
45. Bordag, M., Mohideen, U., Mostepanenko, V.M.: New developments in the Casimir effect. Phys. Rep. **353**, 1–205 (2001). quant-ph/0106045
46. Bordag, M., Falomir, H., Santangelo, E.M., Vassilevich, D.V.: Boundary dynamics and multiple reflection expansion for Robin boundary conditions. Phys. Rev. D **65**, 064032 (2002). hep-th/0111073
47. Bordag, M., Goldhaber, A.S., van Nieuwenhuizen, P., Vassilevich, D.: Heat kernels and zeta-function regularization for the mass of the SUSY kink. Phys. Rev. D **66**, 125014 (2002). hep-th/0203066

48. Bousso, R., Hawking, S.W.: Trace anomaly of dilaton coupled scalars in two dimensions. Phys. Rev. D **56**, 7788–7791 (1997). hep-th/9705236
49. Branson, T.P., Gilkey, P.B.: The asymptotics of the Laplacian on a manifold with boundary. Commun. Partial Differ. Equ. **15**, 245–272 (1990)
50. Branson, T.P., Gilkey, P.B.: Residues of the eta function for an operator of Dirac type with local boundary conditions. Differ. Geom. Appl. **2**, 249 (1992)
51. Branson, T.P., Gilkey, P.B., Vassilevich, D.V.: Vacuum expectation value asymptotics for second order differential operators on manifolds with boundary. J. Math. Phys. **39**, 1040–1049 (1998). hep-th/9702178
52. Branson, T.P., Gilkey, P.B., Kirsten, K., Vassilevich, D.V.: Heat kernel asymptotics with mixed boundary conditions. Nucl. Phys. B **563**, 603–626 (1999). hep-th/9906144
53. Buchbinder, I.L., Odintsov, S.D., Shapiro, I.L.: Effective Action in Quantum Gravity. IOP Publishing, Bristol (1992)
54. Byers, N.: E. Noether's discovery of the deep connection between symmetries and conservation laws (1998). physics/9807044
55. Bytsenko, A.A., Cognola, G., Vanzo, L., Zerbini, S.: Quantum fields and extended objects in space-times with constant curvature spatial section. Phys. Rep. **266**, 1–126 (1996). hep-th/9505061
56. Callan, C.G., Lovelace, C., Nappi, C.R., Yost, S.A.: Loop corrections to superstring equations of motion. Nucl. Phys. B **308**, 221 (1988)
57. Camporesi, R.: Harmonic analysis and propagators on homogeneous spaces. Phys. Rep. **196**, 1–134 (1990)
58. Chamseddine, A.H.: Noncommutative geometry as the key to unlock the secrets of space-time (2009). 0901.0577
59. Chamseddine, A.H., Connes, A.: The spectral action principle. Commun. Math. Phys. **186**, 731–750 (1997). hep-th/9606001
60. Chamseddine, A.H., Connes, A.: Quantum gravity boundary terms from spectral action. Phys. Rev. Lett. **99**, 071302 (2007). 0705.1786
61. Chamseddine, A.H., Connes, A., Marcolli, M.: Gravity and the standard model with neutrino mixing. Adv. Theor. Math. Phys. **11**, 991–1089 (2007). hep-th/0610241
62. Chandrasekhar, S.: The Mathematical Theory of Black Holes, 646 pp. Clarendon, Oxford (1992)
63. Cheeger, J.: Spectral geometry of singular Riemannian spaces. J. Differ. Geom. **18**, 575 (1983)
64. Chepelev, I., Roiban, R.: Renormalization of quantum field theories on noncommutative $R^d$. I: scalars. J. High Energy Phys. **05**, 037 (2000). hep-th/9911098
65. Chernikov, N.A., Tagirov, E.A.: Quantum theory of scalar fields in de Sitter space-time. Ann. Poincaré Phys. Theor. A **9**, 109 (1968)
66. Chodos, A., Jaffe, R.L., Johnson, K., Thorn, C.B.: Baryon structure in the bag theory. Phys. Rev. D **10**, 2599 (1974)
67. Chodos, A., Jaffe, R.L., Johnson, K., Thorn, C.B., Weisskopf, V.F.: A new extended model of hadrons. Phys. Rev. D **9**, 3471–3495 (1974)
68. Christensen, S.M., Fulling, S.A.: Trace anomalies and the Hawking effect. Phys. Rev. D **15**, 2088–2104 (1977)
69. Connes, A.: Noncommutative Geometry. Academic Press, London (1994)
70. Connes, A.: Noncommutative geometry and the standard model with neutrino mixing. J. High Energy Phys. **11**, 081 (2006). hep-th/0608226
71. Connes, A., Marcolli, M.: Noncommutative Geometry, Quantum Fields and Motives. Am. Math. Soc. and Hindustan Book Agency, Providence and Gurgaon (2007)
72. Connes, A., Tretkoff, P.: The Gauss-Bonnet Theorem for the noncommutative two torus. arXiv:0910.0188v1 [math.QA]
73. Dai, J., Leigh, R.G., Polchinski, J.: New connections between string theories. Mod. Phys. Lett. A **4**, 2073–2083 (1989)

74. Dashen, R.F., Hasslacher, B., Neveu, A.: Nonperturbative methods and extended hadron models in field theory. 2. Two-dimensional models and extended hadrons. Phys. Rev. D **10**, 4130–4138 (1974)
75. Deser, S., Jackiw, R.: Classical and quantum scattering on a cone. Commun. Math. Phys. **118**, 495 (1988)
76. Deser, S., Griguolo, L., Seminara, D.: Gauge invariance, finite temperature and parity anomaly in $D = 3$. Phys. Rev. Lett. **79**, 1976–1979 (1997). hep-th/9705052
77. DeWitt, B.S.: Dynamical Theory of Groups and Fields. Gordon & Breach, New York (1965)
78. DeWitt, B.S.: Quantum theory of gravity. 1. The canonical theory. Phys. Rev. **160**, 1113–1148 (1967)
79. DeWitt, B.S.: Quantum theory of gravity. II. The manifestly covariant theory. Phys. Rev. **162**, 1195–1239 (1967)
80. DeWitt, B.S.: Quantum theory of gravity. III. Applications of the covariant theory. Phys. Rev. **162**, 1239–1256 (1967)
81. D'Hoker, E., Jackiw, R.: Liouville field theory. Phys. Rev. D **26**, 3517 (1982)
82. Dilkes, F.A., Duff, M.J., Liu, J.T., Sati, H.: Quantum discontinuity between zero and infinitesimal graviton mass with a Lambda term. Phys. Rev. Lett. **87**, 041301 (2001). hep-th/0102093
83. Donnelly, H.: Spectrum and the fixed point set of isometries I. Math. Ann. **224**, 161 (1976)
84. Doplicher, S., Fredenhagen, K., Roberts, J.E.: Space-time quantization induced by classical gravity. Phys. Lett. B **331**, 39–44 (1994)
85. Doplicher, S., Fredenhagen, K., Roberts, J.E.: The quantum structure of space-time at the Planck scale and quantum fields. Commun. Math. Phys. **172**, 187–220 (1995). hep-th/0303037
86. Douglas, M.R., Nekrasov, N.A.: Noncommutative field theory. Rev. Mod. Phys. **73**, 977–1029 (2001). hep-th/0106048
87. Dowker, J.S.: Quantum field theory on a cone. J. Phys. A **10**, 115–124 (1977)
88. Dowker, J.S.: Vacuum averages for arbitrary spin around a cosmic string. Phys. Rev. D **36**, 3742 (1987)
89. Dowker, J.S.: Effective actions with fixed points. Phys. Rev. D **50**, 6369–6373 (1994). hep-th/9406144
90. Dowker, J.S.: Heat kernels on curved cones. Class. Quantum Gravity **11**, L137–L140 (1994). hep-th/9406002
91. Dowker, J.S.: Two simple topics in spectral theory (2001). hep-th/0107171
92. Dowker, J.S., Critchley, R.: Effective Lagrangian and energy momentum tensor in de Sitter space. Phys. Rev. D **13**, 3224 (1976)
93. Dowker, J.S., Kennedy, G.: Finite temperature and boundary effects in static space-times. J. Phys. A **11**, 895 (1978)
94. Dowker, J.S., Kirsten, K.: The a$(3/2)$ heat kernel coefficient for oblique boundary conditions. Class. Quantum Gravity **16**, 1917–1936 (1999). hep-th/9806168
95. Duff, M.J.: Twenty years of the Weyl anomaly. Class. Quantum Gravity **11**, 1387–1404 (1994). hep-th/9308075
96. Duff, M.J., Liu, J.T., Sati, H.: Quantum $M^2 \to 2\Lambda/3$ discontinuity for massive gravity with a $\Lambda$ term. Phys. Lett. B **516**, 156–160 (2001). hep-th/0105008
97. Duff, M.J., Liu, J.T., Sati, H.: Quantum discontinuity for massive spin $3/2$ with a $\Lambda$ term. Nucl. Phys. B **680**, 117–130 (2004). hep-th/0211183
98. Dunne, G.V.: Aspects of Chern-Simons theory (1998). hep-th/9902115
99. Eguchi, T., Gilkey, P.B., Hanson, A.J.: Gravitation, gauge theories and differential geometry. Phys. Rep. **66**, 213 (1980)
100. Elizalde, E.: Ten Physical Applications of Spectral Zeta Functions. Springer, Berlin (1995)
101. Elizalde, E., Odintsov, S., Romeo, A., Bytsenko, A.A., Zerbini, S.: Zeta Regularization Technique with Applications. World Scientific, Singapore (1994)
102. Elizalde, E., Vanzo, L., Zerbini, S.: Zeta-function regularization, the multiplicative anomaly and the Wodzicki residue. Commun. Math. Phys. **194**, 613–630 (1998). hep-th/9701060
103. Esposito, G.: Dirac Operators and Spectral Geometry. Cambridge University Press, Cambridge (1997)

104. Esposito, G.: Quantum Gravity in Four-Dimensions. Nova Science, New York (2001)
105. Esposito, G., Kamenshchik, A.Y., Pollifrone, G.: Euclidean Quantum Gravity on Manifolds with Boundary. Kluwer Academic, Dordrecht (1997)
106. Faddeev, L.D., Korepin, V.E.: Quantum theory of solitons: preliminary version. Phys. Rep. **42**, 1–87 (1978)
107. Faddeev, L.D., Popov, V.N.: Feynman diagrams for the Yang-Mills field. Phys. Lett. B **25**, 29–30 (1967)
108. Faddeev, L.D., Slavnov, A.A.: Gauge fields. Introduction to quantum theory. Front. Phys. **50**, 1–232 (1980)
109. Feynman, R.P., Hibbs, A.R.: Quantum Mechanics and Path Integrals. McGraw-Hill, New York (1965)
110. Flügge, S.: Practical Quantum Mechanics. Springer, Berlin (1971)
111. Fock, V.A.: The proper time in classical and quantum mechanics. Izv. Akad. Nauk USSR (Phys.) **4–5**, 551–568 (1937)
112. Fradkin, E.S., Tseytlin, A.A.: Nonlinear electrodynamics from quantized strings. Phys. Lett. B **163**, 123 (1985)
113. Frolov, V.P., Fursaev, D.V.: Thermal fields, entropy, and black holes. Class. Quantum Gravity **15**, 2041–2074 (1998). hep-th/9802010
114. Frolov, V., Novikov, I.: Black Hole Physics: Basic Concepts and New Developments. Kluwer Academic, Dordrecht (1998)
115. Frolov, V.P., Fursaev, D.V., Page, D.N.: Thorny spheres and black holes with strings. Phys. Rev. D **65**, 104029 (2002). hep-th/0112129
116. Fujikawa, K.: Path integral measure for gauge invariant fermion theories. Phys. Rev. Lett. **42**, 1195 (1979)
117. Fulling, S.A.: The local geometric asymptotics of continuum eigenfunction expansions I. Overview. SIAM J. Math. Anal. **13**, 891 (1982)
118. Fulling, S.A.: Some properties of Riesz means and spectral expansions. Electr. J. Differ. Equ. **6**, 1–39 (1999). physics/9710006
119. Fulling, S.A., Ruijsenaars, S.N.M.: Temperature, periodicity and horizons. Phys. Rep. **152**, 135 (1987)
120. Fursaev, D.V.: Spectral geometry and one loop divergences on manifolds with conical singularities. Phys. Lett. B **334**, 53–60 (1994). hep-th/9405143
121. Fursaev, D.V.: Euclidean and canonical formulations of statistical mechanics in the presence of Killing horizons. Nucl. Phys. B **524**, 447–468 (1998). hep-th/9709213
122. Fursaev, D.V.: Energy, Hamiltonian, Noether charge, and black holes. Phys. Rev. D **59**, 064020 (1999). hep-th/9809049
123. Fursaev, D.V.: Spectral asymptotics of eigen-value problems with non-linear dependence on the spectral parameter. Class. Quantum Gravity **19**, 3635 (2002)
124. Fursaev, D.V., Miele, G.: Cones, spins and heat kernels. Nucl. Phys. B **484**, 697–723 (1997). hep-th/9605153
125. Gayral, V., Iochum, B.: The spectral action for Moyal planes. J. Math. Phys. **46**, 043503 (2005). hep-th/0402147
126. Gayral, V., Gracia-Bondia, J.M., Iochum, B., Schucker, T., Varilly, J.C.: Moyal planes are spectral triples. Commun. Math. Phys. **246**, 569–623 (2004). hep-th/0307241
127. Gayral, V., Iochum, B., Vassilevich, D.V.: Heat kernel and number theory on NC-torus. Commun. Math. Phys. **273**, 415–443 (2007). hep-th/0607078
128. Gel'fand, I.M., Shilov, G.E.: Generalized Functions, vol. 1. Academic Press, New York (1964)
129. Gibbons, G.W.: In: General Relativity: An Einstein Centenary Survey, 639 pp. Cambridge University Press, Cambridge (1979)
130. Gibbons, G.W., Hawking, S.W.: Action integrals and partition functions in quantum gravity. Phys. Rev. D **15**, 2752–2756 (1977)
131. Gibbons, G.W., Hawking, S.W.: Cosmological event horizons, thermodynamics, and particle creation. Phys. Rev. D **15**, 2738 (1977)

132. Gilkey, P.B.: The spectral geometry of a Riemannian manifold. J. Differ. Geom. **10**, 601–618 (1975)
133. Gilkey, P.B.: Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem. CRC Press, Boca Raton (1994)
134. Gilkey, P.B.: Asymptotic Formulae in Spectral Geometry. CRC Press, Boca Raton (2004)
135. Gilkey, P.B., Kirsten, K., Vassilevich, D.V.: Heat trace asymptotics with transmittal boundary conditions and quantum brane-world scenario. Nucl. Phys. B **601**, 125–148 (2001). hep-th/0101105
136. Gilkey, P.B., Kirsten, K., Vassilevich, D.: Heat trace asymptotics defined by transfer boundary conditions. Lett. Math. Phys. **63**, 29–37 (2003). hep-th/0208130
137. Gitman, D.M., Tyutin, I.V.: Quantization of Fields with Constraints. Springer, Berlin (1990)
138. Goldstone, J., Jackiw, R.: Quantization of nonlinear waves. Phys. Rev. D **11**, 1486–1498 (1975)
139. Goncalves, B., de Berredo-Peixoto, G., Shapiro, I.L.: Exact formfactors in the one-loop curved-space QED and the nonlocal multiplicative anomaly. Int. J. Mod. Phys. A **25**, 2382–2390 (2010). 0904.4171
140. Gracia-Bondia, J.M., Varilly, J.C.: Algebras of distributions suitable for phase space quantum mechanics. 1. J. Math. Phys. **29**, 869–879 (1988)
141. Gracia-Bondia, J.M., Varilly, J.C., Figueroa, H.: Elements of Noncommutative Geometry. Birkhäuser, Boston (2001)
142. Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products. Academic Press, New York (1994)
143. Graham, N., Jaffe, R.L.: Energy, central charge, and the BPS bound for $1+1$ dimensional supersymmetric solitons. Nucl. Phys. B **544**, 432–447 (1999). hep-th/9808140
144. Green, M.B., Schwarz, J.H., Witten, E.: Superstring Theories, vols. 1 and 2. Cambridge University Press, Cambridge (1987)
145. Gross, D.J., Wilczek, F.: Ultraviolet behavior of non-abelian gauge theories. Phys. Rev. Lett. **30**, 1343–1346 (1973)
146. Grosse, H., Wulkenhaar, R.: Renormalisation of $\phi^4$ theory on noncommutative $R^4$ in the matrix base. Commun. Math. Phys. **256**, 305–374 (2005). hep-th/0401128
147. Grubb, G.: Properties of normal boundary problems for elliptic even-order systems. Ann. Sc. Norm. Super. Pisa **1**, 1 (1974)
148. Grubb, G.: Functional Calculus of Pseudo-differential Boundary Problems. Birkhäuser, Boston (1996)
149. Grumiller, D., Kummer, W., Vassilevich, D.V.: Dilaton gravity in two dimensions. Phys. Rep. **369**, 327–430 (2002). hep-th/0204253
150. Gurau, R., Magnen, J., Rivasseau, V., Tanasa, A.: A translation-invariant renormalizable noncommutative scalar model. Commun. Math. Phys. **287**, 275–290 (2009). 0802.0791
151. Gusev, Y.V., Zelnikov, A.I.: Finite temperature nonlocal effective action for quantum fields in curved space. Phys. Rev. D **59**, 024002 (1999). hep-th/9807038
152. Gusev, Y.V., Zelnikov, A.I.: Two-dimensional effective action for matter fields coupled to the dilaton. Phys. Rev. D **61**, 084010 (2000). hep-th/9910198
153. Hansson, T.H., Jaffe, R.L.: The multiple reflection expansion for confined scalar, Dirac and gauge fields. Ann. Phys. **151**, 204 (1983)
154. Hartle, J.B., Hawking, S.W.: Path integral derivation of black hole radiance. Phys. Rev. D **13**, 2188 (1976)
155. Hawking, S.W.: Zeta function regularization of path integrals in curved space-time. Commun. Math. Phys. **55**, 133 (1977)
156. Hawking, S.W., Ellis, C.F.R.: The Large Scale Structure of Space-Time. Cambridge University Press, Cambridge (1973)
157. Hawking, S.W., Reall, H.S.: Charged and rotating AdS black holes and their CFT duals. Phys. Rev. D **61**, 024014 (2000). hep-th/9908109
158. Henneaux, M.: Hamiltonian form of the path integral for theories with a gauge freedom. Phys. Rep. **126**, 1–66 (1985)

159. Higson, N.: The Residue Index Theorem of Connes and Moscovici. In: Roe, J., Higson, N. (eds.) Surveys in Noncommutative Geometry. Clay Mathematics Proceedings, pp. 71–126. Am. Math. Soc., Providence (2006)
160. Hod, S.: Bohr's correspondence principle and the area spectrum of quantum black holes. Phys. Rev. Lett. **81**, 4293 (1998). gr-qc/9812002
161. Huang, K.: Quarks, Leptons and Gauge Fields. World Scientific, Singapore (1982)
162. Israel, W.: Singular hypersurfaces and thin shells in general relativity. Nuovo Cimento B **44S10**, 1 (1966)
163. Itzykson, C., Zuber, J.-B.: Quantum Field Theory. McGraw-Hill, New York (1980)
164. Ivey, T.A., Landsberg, J.M.: Cartan for Beginners. Am. Math. Soc., Providence (2003)
165. Kac, M.: Can one hear the shape of a drum? Am. Math. Mon. **73**, 1–23 (1966)
166. Kapusta, J.I., Gale, C.: Finite-temperature Field Theory: Principles and Applications. Cambridge University Press, Cambridge (2006)
167. Kay, B.S., Studer, U.M.: Boundary conditions for quantum mechanics on cones and fields around cosmic strings. Commun. Math. Phys. **139**, 103–140 (1991)
168. Keldysh, M.V.: On eigen-values and eigen-functions of some classes of non-self-adjoint operators. Dokl. Akad. Nauk SSSR **77**, 11 (1951) (in Russian, see English translation in the book by A.S. Markus)
169. Kirsten, K.: Spectral Functions in Mathematics and Physics. Chapman & Hall/CRC, Boca Raton (2001)
170. Kirsten, K., Loya, P.: Computation of determinants using contour integrals. Am. J. Phys. **76**, 60–64 (2008). 0707.3755
171. Kokkotas, K.D., Schmidt, B.G.: Quasi-normal modes of stars and black holes. Living Rev. Relativ. **2**, 2 (1999). gr-qc/9909058
172. Konoplya, R.A., Vassilevich, D.V.: Quantum corrections to noncommutative kink. J. High Energy Phys. **01**, 068 (2008). 0712.0360
173. Konoplya, R.A., Zhidenko, A.: Quasinormal modes of black holes: from astrophysics to string theory (2011). 1102.4014
174. Kontsevich, M., Vishik, S.: Determinants of elliptic pseudodifferential operators (1994). hep-th/9404046
175. Kühnel, W.: Differential Geometry. Curves-Surfaces-Manifolds. Am. Math. Soc., Providence (2002)
176. Kummer, W., Vassilevich, D.V.: Renormalizability of the open string sigma model and emergence of d-branes. J. High Energy Phys. **07**, 012 (2000). hep-th/0006108
177. Landsman, N.P., van Weert, C.G.: Real and imaginary time field theory at finite temperature and density. Phys. Rep. **145**, 141 (1987)
178. Leigh, R.G.: Dirac-Born-Infeld action from Dirichlet sigma model. Mod. Phys. Lett. A **4**, 2767 (1989)
179. Leutwyler, H.: Gravitational anomalies: a soluble two-dimensional model. Phys. Lett. B **153**, 65 (1985)
180. Leutwyler, H., Mallik, S.: Gravitational anomalies. Z. Phys. C **33**, 205 (1986)
181. Luckock, H.: Mixed boundary conditions in quantum field theory. J. Math. Phys. **32**, 1755 (1991)
182. Markus, A.S.: Introduction to the Spectral Theory of Polynomial Operator Pencils. Translations of Mathematical Monographs, vol. 71. Am. Math. Soc., Providence (1988)
183. Martin, P.C., Schwinger, J.S.: Theory of many particle systems. I. Phys. Rev. **115**, 1342–1373 (1959)
184. Matsubara, T.: A new approach to quantum statistical mechanics. Prog. Theor. Phys. **14**, 351–378 (1955)
185. McAvity, D.M., Osborn, H.: A DeWitt expansion of the heat kernel for manifolds with a boundary. Class. Quantum Gravity **8**, 603–638 (1991)
186. McAvity, D.M., Osborn, H.: Asymptotic expansion of the heat kernel for generalized boundary conditions. Class. Quantum Gravity **8**, 1445–1454 (1991)
187. Milton, K.A.: The Casimir Effect: Physical Manifestations of Zero-Point Energy. World Scientific, River Edge (2001)

188. Milton, K.A.: The Casimir effect: Recent controversies and progress. J. Phys. A **37**, R209 (2004). hep-th/0406024
189. Minakshisundaram, S., Pleijel, A.: Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds. Can. J. Math. **1**, 242–256 (1949)
190. Minwalla, S., Van Raamsdonk, M., Seiberg, N.: Noncommutative perturbative dynamics. J. High Energy Phys. **02**, 020 (2000). hep-th/9912072
191. Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation. Freeman, San Francisco (1973)
192. Morozov, A.Y.: Anomalies in gauge theories. Sov. Phys. Usp. **29**, 993–1039 (1986)
193. Moss, I.G.: Quantum Theory, Black Holes and Inflation. Wiley, New York (1996)
194. Mukhanov, V.F., Winitzki, S.: Introduction to Quantum Effects in Gravity. Cambridge University Press, Cambridge (2007)
195. Naimark, M.A.: Lineare Differentialoperatoren. Akademie Verlag, Berlin (1960)
196. Nakahara, M.: Geometry, Topology and Physics. Hilger, Bristol (1990)
197. Nastase, H., Stephanov, M.A., van Nieuwenhuizen, P., Rebhan, A.: Topological boundary conditions, the BPS bound, and elimination of ambiguities in the quantum mass of solitons. Nucl. Phys. B **542**, 471–514 (1999). hep-th/9802074
198. Nesterov, D., Solodukhin, S.N.: Gravitational effective action and entanglement entropy in UV modified theories with and without Lorentz symmetry. Nucl. Phys. B **842**, 141–171 (2011). 1007.1246
199. Niemi, A.J.: Topological terms induced by finite temperature and density fluctuations. Phys. Rev. Lett. **57**, 1102–1105 (1986)
200. Niemi, A.J., Semenoff, G.W.: A comment on ‘induced Chern-Simons terms at high temperatures and finite densities’. Phys. Rev. Lett. **54**, 2166 (1985)
201. Novozhilov, Y.V., Vassilevich, D.V.: Induced classical gravity. Lett. Math. Phys. **21**, 253–271 (1991)
202. Oldham, K.B., Spanier, J.: The Fractional Calculus. Academic Press, New York (1974)
203. Osborn, H.: Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories. Nucl. Phys. B **363**, 486–526 (1991)
204. Peebles, P.J.E., Ratra, B.: The cosmological constant and dark energy. Rev. Mod. Phys. **75**, 559–606 (2003). astro-ph/0207347
205. Peskin, M.E., Schroeder, D.V.: An Introduction to Quantum Field Theory. Addison-Wesley, Reading (1995)
206. Piacitelli, G.: Quantum spacetime: a disambiguation. SIGMA **6**, 073 (2010). 1004.5261
207. Polchinski, J.: Dirichlet-branes and ramond-ramond charges. Phys. Rev. Lett. **75**, 4724–4727 (1995). hep-th/9510017
208. Polchinski, J.: String Theory. Cambridge University Press, Cambridge (1998)
209. Politzer, H.D.: Reliable perturbative results for strong interactions? Phys. Rev. Lett. **30**, 1346–1349 (1973)
210. Prasad, M.K., Sommerfield, C.M.: An exact classical solution for the ’t Hooft monopole and the Julia-Zee dyon. Phys. Rev. Lett. **35**, 760–762 (1975)
211. Rajaraman, R.: Solitons and Instantons. Elsevier, Amsterdam (1996)
212. Randall, L., Sundrum, R.: A large mass hierarchy from a small extra dimension. Phys. Rev. Lett. **83**, 3370–3373 (1999). hep-ph/9905221
213. Randall, L., Sundrum, R.: An alternative to compactification. Phys. Rev. Lett. **83**, 4690–4693 (1999). hep-th/9906064
214. Rebhan, A., van Nieuwenhuizen, P.: No saturation of the quantum Bogomolnyi bound by two-dimensional $N = 1$ supersymmetric solitons. Nucl. Phys. B **508**, 449–467 (1997). hep-th/9707163
215. Rebhan, A., van Nieuwenhuizen, P., Wimmer, R.: New developments in the quantization of supersymmetric solitons (kinks, vortices and monopoles). Braz. J. Phys. **34**, 1273–1287 (2004). hep-th/0404223
216. Redlich, A.N.: Gauge noninvariance and parity nonconservation of three-dimensional fermions. Phys. Rev. Lett. **52**, 18 (1984)
217. Redlich, A.N.: Parity violation and gauge noninvariance of the effective gauge field action in three-dimensions. Phys. Rev. D **29**, 2366–2374 (1984)

218. Riesz, F., Szokefalvi-Nagy, B.: Functional Analysis. Ungar, New York (1955)
219. Robinson, S.P., Wilczek, F.: A relationship between Hawking radiation and gravitational anomalies. Phys. Rev. Lett. **95**, 011303 (2005). gr-qc/0502074
220. Rubakov, V.A., Shaposhnikov, M.E.: Do we live inside a domain wall? Phys. Lett. B **125**, 136–138 (1983)
221. Sakharov, A.D.: Spectral density of eigenvalues of the wave equation and the vacuum polarization. Theor. Math. Phys. **23**, 435–444 (1976)
222. Schleich, K., Witt, D.M.: Generalized sums over histories for quantum gravity. 1. Smooth conifolds. Nucl. Phys. B **402**, 411–468 (1993). gr-qc/9307015
223. Schomerus, V.: D-branes and deformation quantization. J. High Energy Phys. **06**, 030 (1999). hep-th/9903205
224. Schutz, B.F.: Geometrical Methods of Mathematical Physics. Cambridge University Press, Cambridge (1982)
225. Schwinger, J.: On gauge invariance and vacuum polarization. Phys. Rev. **82**, 664–679 (1951)
226. Seeley, R.: Singular integrals and boundary value problems. Am. J. Math. **88**, 781–809 (1966)
227. Seeley, R.T.: Complex powers of an elliptic operator. In: Singular Integrals, Chicago 1966. Proc. Symp. Pure Math., vol. 10, pp. 308–315. Am. Math. Soc., Providence (1968)
228. Seeley, R.: The resolvent of an elliptic boundary value problem. Am. J. Math. **91**, 889–920 (1969)
229. Seiberg, N.: Notes on quantum Liouville theory and quantum gravity. Prog. Theor. Phys. Suppl. **102**, 319–349 (1990)
230. Seiberg, N., Witten, E.: String theory and noncommutative geometry. J. High Energy Phys. **09**, 032 (1999). hep-th/9908142
231. Shifman, M., Yung, A.: Supersymmetric solitons and how they help us understand non-abelian gauge theories. Rev. Mod. Phys. **79**, 1139 (2007). hep-th/0703267
232. Shifman, M.A., Vainshtein, A.I., Voloshin, M.B.: Anomaly and quantum corrections to solitons in two-dimensional theories with minimal supersymmetry. Phys. Rev. D **59**, 045016 (1999). hep-th/9810068
233. Shubin, M.A.: Pseudodifferential Operators and Spectral Theory. Springer, Berlin (1987)
234. Sommerfeld, A.: Über verzweigte Potentiale im Raum. Proc. Lond. Math. Soc. **28**, 395 (1897)
235. Synge, J.L.: Relativity: The General Theory. North-Holland, Amsterdam (1960)
236. Szabo, R.J.: Quantum field theory on noncommutative spaces. Phys. Rep. **378**, 207–299 (2003). hep-th/0109162
237. Titchmarsh, E.C.: Eigenfunction Expansions Associated with Second-Order Differential Equations. Clarendon, Oxford (1958)
238. Townsend, P.K.: Black holes (1997). gr-qc/9707012
239. Tseytlin, A.A.: Vector field effective action in the open superstring theory. Nucl. Phys. B **276**, 391 (1986)
240. van Nieuwenhuizen, P., Waldron, A.: On Euclidean spinors and Wick rotations. Phys. Lett. B **389**, 29–36 (1996). hep-th/9608174
241. Vasil'ev, A.N.: The Field Theoretic Renormalization Group in Critical Behavior Theory and Stochastic Dynamics. Chapman & Hall/CRC, Boca Raton (2004)
242. Vassilevich, D.V.: QED on curved background and on manifolds with boundaries: unitarity versus covariance. Phys. Rev. D **52**, 999–1010 (1995). gr-qc/9411036
243. Vassilevich, D.V.: Heat kernel expansion: User's manual. Phys. Rep. **388**, 279–360 (2003). hep-th/0306138
244. Vassilevich, D.V.: Non-commutative heat kernel. Lett. Math. Phys. **67**, 185–194 (2004). hep-th/0310144
245. Vassilevich, D.V.: Quantum noncommutative gravity in two dimensions. Nucl. Phys. B **715**, 695–712 (2005). hep-th/0406163
246. Vassilevich, D.V.: Heat kernel, effective action and anomalies in noncommutative theories. J. High Energy Phys. **08**, 085 (2005). hep-th/0507123

247. Vassilevich, D.V.: Induced Chern-Simons action on noncommutative torus. Mod. Phys. Lett. A **22**, 1255–1263 (2007). hep-th/0701017
248. Vassilevich, D.V.: Heat trace asymptotics on noncommutative spaces. SIGMA **3**, 093 (2007). 0708.4209
249. Vassilevich, D.V., Yurov, A.: Space-time non-commutativity tends to create bound states. Phys. Rev. D **69**, 105006 (2004). hep-th/0311214
250. Vilenkin, A., Schellard, E.P.S.: Cosmic Strings and Other Topological Defects. Cambridge University Press, Cambridge (2000)
251. Wald, R.M.: General Relativity. University of Chicago Press, Chicago (1984)
252. Weinberg, S.: Gravitation and Cosmology. Wiley, New York (1972)
253. Weinberg, S.: The Quantum Theory of Fields. Vol. 1: Foundations. Cambridge University Press, Cambridge (1995)
254. Wess, J., Bagger, J.: Supersymmetry and Supergravity. Princeton University Press, Princeton (1992)
255. Wess, J., Zumino, B.: Consequences of anomalous Ward identities. Phys. Lett. B **37**, 95 (1971)
256. West, P.C.: Introduction to Supersymmetry and Supergravity. World Scientific, Singapore (1990)
257. Weyl, H.: Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieler Differentialgleichungen. Math. Ann. **71**, 441–479 (1912)
258. Weyl, H.: Das asymptotische Verteilungsgesetz der Eigenschwingungen eines beliebig gestalteten elastischen Körpers. Rend. Circ. Mat. Palermo **39**, 1–50 (1915)
259. Witten, E., Olive, D.I.: Supersymmetry algebras that include topological charges. Phys. Lett. B **78**, 97 (1978)
260. York, J.W.: Role of conformal three geometry in the dynamics of gravitation. Phys. Rev. Lett. **28**, 1082–1085 (1972)
261. Zeldovich, Y.B.: Cosmological constant and elementary particles. JETP Lett. **6**, 316 (1967)

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